

On the Lagrange and Markov Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature.

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Abstract

We consider the Lagrange and the Markov dynamical spectra associated to a geodesic flow on a surface of negative curvature. We show that for a large set of real functions on the unit tangent bundle and for typical metrics with negative curvature and finite volume, both the Lagrange and the Markov dynamical spectra have non-empty interior.

1 Introduction

A mathematical object closely related to our work is the classical *Lagrange spectrum* (cf. [CF89]), which we describe in the following: Given an irrational number α , according to Dirichlet's theorem the inequality $\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$ has infinitely many rational solutions $\frac{p}{q}$. Markov and Hurwitz improved this result (cf. [CF89]), proving that, for all irrational α , the inequality $\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}$ has infinitely many rational solutions $\frac{p}{q}$. This is the best result which holds for all irrational numbers α : for $\alpha = \frac{1+\sqrt{5}}{2}$, for instance, the constant $\sqrt{5}$ in the denominator of the inequality cannot be improved.

Meanwhile, for a fixed irrational α better results can be expected. We associate, to each α , its best constant of approximation (Lagrange value of α), given by

$$\begin{aligned} k(\alpha) &= \sup \left\{ k > 0 : \left| \alpha - \frac{p}{q} \right| < \frac{1}{kq^2} \text{ has infinite rational solutions } \frac{p}{q} \right\} \\ &= \limsup_{\substack{p,q \rightarrow \infty \\ p,q \in \mathbb{N}}} |q(q\alpha - p)|^{-1} \in \mathbb{R} \cup \{+\infty\}. \end{aligned}$$

Then, it always holds that $k(\alpha) \geq \sqrt{5}$. The set

$$L = \{k(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ and } k(\alpha) < \infty\}$$

is known as the *Lagrange spectrum*.

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Regular Cantor sets on the line play a fundamental role in dynamical systems. They are defined by expansive maps and have some kind of self similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion (see precise definition in section 5). Some background on regular Cantor sets which is relevant to our work can be found in [CF89], [PT93], [MY01] and [MY10].

In 1947 M. Hall (cf. [Hal47]) proved that the regular Cantor set $C(4)$ of real numbers in $[0, 1]$ in whose continued fraction only appear coefficients 1, 2, 3, 4 satisfies

$$C(4) + C(4) = [\sqrt{2} - 1, 4(\sqrt{2} - 1)].$$

Let α irrational expressed in continued fractions by $\alpha = [a_0; a_1, \dots]$, for $n \in \mathbb{N}$. Defining $\alpha_n = [a_n; a_{n+1}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots, a_1]$, it can be proved by elementary techniques that

$$k(\alpha) = \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n).$$

With this latter characterization of the Lagrange spectrum and from Hall's result it follows that $L \supset [6, +\infty)$, so the Lagrange spectrum contains a whole half-line - such a half-line is known as a *Hall's ray* of the Lagrange spectrum.

In 1975, G. Freiman (cf. [Fre75] and [CF89]) proved some difficult results showing that the arithmetic sum of certain (regular) Cantor sets, related to continued fractions contain intervals, and used them to determined the precise beginning of Hall's ray (the biggest half-line contained in L), which is

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616 \dots$$

Another interesting set related to diophantine approximations is the classical *Markov spectrum* defined by (cf. [CF89])

$$M = \left\{ \inf_{(x,y) \in \mathbb{Z}^2 \setminus (0,0)} |f(x,y)|^{-1} : f(x,y) = ax^2 + bxy + cy^2 \text{ with } b^2 - 4ac = 1 \right\}. \quad (1)$$

Both the Lagrange and Markov spectrum have a dynamical interpretation, that is of interest for our work.

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ and $\sigma: \Sigma \rightarrow \Sigma$ the shift defined by $\sigma((a_n)_{n \in \mathbb{Z}}) = (a_{n+1})_{n \in \mathbb{Z}}$. If $f: \Sigma \rightarrow \mathbb{R}$ is defined by $f((a_n)_{n \in \mathbb{Z}}) = \alpha_0 + \beta_0 = [a_0, a_1, \dots] + [0, a_{-1}, a_{-2}, \dots]$, then

$$L = \left\{ \limsup_{n \rightarrow \infty} f(\sigma^n(\underline{\theta})) : \underline{\theta} \in \Sigma \right\}$$

and

$$M = \left\{ \sup_{n \in \mathbb{Z}} f(\sigma^n(\underline{\theta})) : \underline{\theta} \in \Sigma \right\}.$$

There is also a geometric interpretation of the Lagrange spectrum which is the main focus of our work (cf. [CF89]). Consider the modular group, $SL(2, \mathbb{Z})$, that is, the set of all 2×2 integer matrices with determinant equal to one, and $PSL(2, \mathbb{Z})$ the projectivization

of $SL(2, \mathbb{Z})$. Given any $V \in SL(2, \mathbb{Z})$, $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define the associated transformation by $V(z) = \frac{az+b}{cz+d}$. Note that if $W = \lambda V$ with $\lambda \in \mathbb{Z}^*$, then $V(z) = W(z)$.

Remember that for an irrational number α the Lagrange value of α is

$$k(\alpha) = \sup\{k : |q(q\alpha - p)| \leq k^{-1} \text{ for infinitely many pairs of positive integers } (p, q)\}.$$

We note that in the above definition we may assume that the positive integers p, q are coprime. In this case there exist integers p', q' such that $q'p - p'q = 1$, so for $V = \begin{pmatrix} q' & -p' \\ -q & p \end{pmatrix} \in SL(2, \mathbb{Z})$ and $V(z) = \frac{q'z - p'}{-qz + p}$, we have

$$k(\alpha) = \sup\{k : |V(\infty) - V(\alpha)|^{-1} = |q(q\alpha - p)| \leq k^{-1} \text{ for infinitely many } V \in SL(2, \mathbb{Z})\}.$$

Let \mathbb{H}^2 be of upper half-plane model of the real hyperbolic plane, with the Poincaré metric, and let $N := \mathbb{H}^2 / PSL(2, \mathbb{Z})$ the modular orbifold. Let e be an end of N (cf. [HP02] and [PP10]), define *the asymptotic height spectrum of the pair* (N, e) by

$$LimsupSp(N, e) = \left\{ \limsup_{t \rightarrow \infty} ht_e(\gamma(t)) : \gamma \in SN \right\}$$

where ht_e is the height associated to the end e of N , defined by

$$ht_e(x) = \lim_{t \rightarrow +\infty} d(x, \Gamma(t)) - t,$$

being Γ a ray that defines the end e , and SN denotes the unitary tangent bundle of N .

Using the latter interpretation of the Lagrange spectrum, the asymptotic height spectrum $LimsupSp(N, e)$ of the modular orbifold N is the image of the Lagrange spectrum by the map $t \rightarrow \log \frac{t}{2}$ (see for instance [[HP02], theorem 3.4]). The geometric interpretation of Freiman's result in our context is that $LimsupSp(N, e)$ contains the maximal interval $[\mu, +\infty)$ with

$$\mu = \log \left(\frac{2221564096 + 283748\sqrt{462}}{2 \cdot 491993569} \right) \approx 0,817095519650396598\dots$$

In 1986, similar results were obtained by A. Haas and C. Series (cf. [HS86]) to the quotient of \mathbb{H}^2 by a fuchsian group of $SL(2, \mathbb{R})$. In particular by the Hecke group G_q defined by

$$G_q = \left\langle \begin{pmatrix} 1 & 2 \cos \pi/q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \text{ for } q \geq 3.$$

In the same year, Andrew Haas [Haa86] obtained results in this direction for hyperbolic Riemann surfaces. Then 11 years later, in 1997, Thomas A. Schmidt and Mark Sheigorn (cf. [SS97]) proved that Riemann surfaces have a Hall's ray in every cusp. In 2012, P. Hubert, L. Marchese and C. Ulcigrai (cf.[HMU12]) showed the existence of Hall's ray in the context of Teichmüller dynamics, more precisely for moduli surfaces, using renormalization. Recently, in 2014, M. Artigiani, L. Marchese, C. Ulcigrai (cf. [AMU14]) showed than Veech surfaces also have a Hall's ray.

Observe that all results mentioned above are on surfaces, which in the geometrical cases have all negative constant curvatures; let us see some known results in dimension greater than or equal to 3, for generalizations of both the Lagrange and Markov spectra.

We may consider the following natural generalization of the Markov spectrum:

Let $B(x) = \sum_{1 \leq i, j \leq n} b_{ij} x_i x_j$, $b_{ij} = b_{ji}$ be a real non-degenerate indefinite quadratic form in n variables and let us denote by Φ_n the set of all such forms. Let $d(B)$ denote the determinant of the matrix (b_{ij}) . Let us set

$$m(B) = \inf_{x \in \mathbb{Z}^n \setminus \{0\}} |B(x)| \quad \text{and} \quad \mu(B) = \frac{m(B)^n}{|d(B)|}.$$

Let M_n denote the set $\mu(\Phi_n)$. G. Margulis in [Mar] showed that for any $n \geq 3$ and $\epsilon > 0$, then the set $M_n \cap (\epsilon, +\infty)$ is a finite set. Since the Lagrange spectrum L , satisfies that $L \subset M$ (cf. [CF89]), then M contains the Hall's ray, but by the foregoing and (1) implies this phenomenon only happens in $n = 2$.

Returning to the geometrical questions, let M be a complete connected Riemannian manifold with sectional curvature at most -1 and let e be an end of it; the associated *Lagrange and Markov Spectra* are defined respectively by

$$LimsupSp(M, e) = \left\{ \limsup_{t \rightarrow \infty} ht_e(\gamma(t)) : \gamma \in SM \right\}$$

and

$$MaxSp(M, e) = \left\{ \sup_{t \in \mathbb{R}} ht_e(\gamma(t)) : \gamma \in SM \right\},$$

where $\gamma(t)$ is the geodesic such that $\gamma(0) = \gamma \in SM$.

In this case, J. Parkkonen and F. Paulin [PP10], using purely geometric arguments showed the following theorems:

Theorem[PP10] *If M has finite volume, dimension $n \geq 3$ and e is an end of M , then $MaxSp(M, e)$ contains the interval $[4.2, +\infty]$.*

Schmidt and Sheingorn [SS97] proved the two-dimensional analogue of the above Theorem in constant curvature -1 . They showed that the maximum height spectrum of a finite area hyperbolic surfaces with respect to any cusp contains the interval $[4.61, +\infty]$.

Theorem[PP10](The Ubiquity of Hall's rays) *If M has finite volume, dimension $n \geq 3$ and e is an end of M , then $LimsupSp(M, e)$ contains the interval $[6.8, +\infty]$.*

These last two theorems can be true in the constant negative curvature 2-dimensional case, but in [[PP10] page 278] J. Parkkonen and F. Paulin expected them to be false in variable curvature and dimension 2.

This paper is inspired in this last question: is it possible that the two previous theorems hold for variable negative curvature in the 2-dimensional case? We prove some positive results in this direction, showing that these spectra have typically non-empty interior, in variable negative curvature and dimension 2.

More precisely, let M be a complete noncompact surface M with metric $\langle \cdot, \cdot \rangle$ such that the Gaussian curvature is bounded between two negative constants and the Gaussian volume is finite: denoting by K_M the Gaussian curvature, we assume that there are constants $a, b > 0$ such that

$$-a^2 \leq K_M \leq -b^2 < 0.$$

From now on we will consider M a surface as above.

Let X be a vector field in $\mathfrak{X}^1(SM)$ and let f be a real function in $C^0(SM, \mathbb{R})$. The dynamical Markov spectrum associated to (f, X) is defined by

$$M(f, X) = \left\{ \sup_{t \in \mathbb{R}} f(X^t(x)) : x \in SM \right\}$$

and the dynamical Lagrange spectrum associated to (f, X) by

$$L(f, X) = \left\{ \limsup_{t \rightarrow \infty} f(X^t(x)) : x \in SM \right\}$$

where $X^t(x)$ is the integral curve of the vector field X in x .

Let ϕ be the vector field in SM defining the geodesic flow of the metric $\langle \cdot, \cdot \rangle$ (here SM denotes the unitary tangent bundle of M). Let $\mathfrak{X}^1(SM)$ denote the space of C^1 vector field on SM .

Theorem 1. *Arbitrarily close to ϕ there is an open set $\mathcal{V} \subset \mathfrak{X}^1(SM)$ such that for any $X \in \mathcal{V}$ we have*

$$\text{int } M(f, X) \neq \emptyset \text{ and } \text{int } L(f, X) \neq \emptyset$$

for any f in a dense and C^2 -open subset \mathcal{U}_X of $C^2(SM, \mathbb{R})$. Moreover, the above statement holds persistently: for any $Y \in \mathcal{V}$, it holds for any (f, X) in a suitable neighborhood of $\mathcal{U}_Y \times \{Y\}$ in $C^2(SM, \mathbb{R}) \times \mathfrak{X}^1(SM)$.

Observe that, in the paper of J. Parkkonen and F. Paulin [PP10], the definition $MaxSp(M, e)$ and $LimSupSp(M, e)$ coincides with $M(f, X)$ and $L(f, X)$, when $f = ht_e \circ \pi$ is the height function ht_e associated to the end e composed with the canonical projection $\pi : SM \rightarrow M$, and X is the vector field that generates the geodesic flow of M .

We also prove a version of Theorem 1 for the restricted case of compositions of functions on the manifold M with the canonical projection.

Theorem 2. *Arbitrarily close to ϕ there is an open set $\mathcal{V} \subset \mathfrak{X}^1(SM)$ such that for any $X \in \mathcal{V}$ we have*

$$\text{int } M(f \circ \pi, X) \neq \emptyset \text{ and } \text{int } L(f \circ \pi, X) \neq \emptyset$$

for any f in a dense and C^2 -open subset $\tilde{\mathcal{U}}_X$ of $C^2(M, \mathbb{R})$. Moreover, the above statement holds persistently: for any $Y \in \mathcal{V}$, it holds for any (f, X) in a suitable neighborhood of $\tilde{\mathcal{U}}_Y \times \{Y\}$ in $C^2(M, \mathbb{R}) \times \mathfrak{X}^1(SM)$.

The previous results can be extended to the following theorems, which requires more sophisticated techniques of perturbations of Riemannian metrics:

Main Theorem 1: *Let M be as above. There is an open set \mathcal{G} of metrics close to $\langle \cdot, \cdot \rangle$ such that, for any $g \in \mathcal{G}$, there is a dense and C^2 -open subset $\mathcal{H}_g \subset C^2(S^g M, \mathbb{R})$ such that*

$$\text{int } M(f, \phi_g) \neq \emptyset \quad \text{and} \quad \text{int } L(f, \phi_g) \neq \emptyset \quad \text{for all } f \in \mathcal{H}_g,$$

where ϕ_g is the vector field defining the geodesic flow of the metric g and $S^g M$ is the unitary tangent bundle of the metric g .

Main Theorem 2: *Let M be as above. There is an open set \mathcal{G} of metrics close to $\langle \cdot, \cdot \rangle$ such that, for any $g \in \mathcal{G}$, there is a dense and C^2 -open subset $\tilde{\mathcal{H}}_g \subset C^2(M, \mathbb{R})$ such that*

$$\text{int } M(f \circ \pi, \phi_g) \neq \emptyset \quad \text{and} \quad \text{int } L(f \circ \pi, \phi_g) \neq \emptyset \quad \text{for all } f \in \tilde{\mathcal{H}}_g.$$

The statements of these two Theorems hold persistently in (f, ϕ_g) , as before.

The problem of finding intervals in the classical Lagrange and Markov spectra is closely related to the study of the fractal geometry of regular Cantor sets related to the Gauss map. However, in the subsequent works on geometrical generalizations of the classical Markov and Lagrange spectra we mentioned above the techniques do not involve fractal geometry or the study of regular Cantor sets. In the present study of two-dimensional spectra, recent results on fractal geometry of regular Cantor sets are (again) a key ingredient in the proofs of our results about dynamical Lagrange and Markov spectra associated to geodesic flows in negative curvature. We use and adapt in this work techniques from [MY01], [MY10] and [MRn13].

The paper is organized as follows: In section 2, we recall some classical results of hyperbolic dynamics which are relevant to this work. In section 3, we construct a hyperbolic set for the geodesic flow, with Hausdorff dimension close to 3. Using this hyperbolic set, we construct a finite number of (disjoint)transversal sections to the geodesic flow, and we show that the Poincaré (first return) map of the union of sections has a hyperbolic invariant set - a horseshoe -with Hausdorff dimension close to 2. In section 4, using the results of section 3, [MRn13], [MY01], [MY10] and some combinatorial techniques (subsection 4.1.3), we prove the Theorems 1 and 2. In subsection 4.2, we develop techniques of perturbations of Riemannian metrics together with further combinatorial techniques in order to adapt constructions of [MY10] in the context of our work, which allows us to adapt the proof of Theorems 1 and 2 in the restricted context of geodesic flows, and obtain the Main Theorems 1 and 2.

2 Preliminaries

A C^1 -flow $\varphi^t: M \rightarrow M$ on a manifold M is said to be an *Anosov flow* if M is hyperbolic set for φ^t (cf. Appendix 5.1).

The central example of Anosov flows is provided by geodesic flows. Given a Riemannian manifold M , denoted by TM the tangent bundle and $SM = \{(x, v) \in TM : \|v\| = 1\}$ the unit tangent bundle of M .

A classic result due to D. Anosov (cf. [Ano69], [Kli82] and [KH95]) states that for complete manifolds of curvature bounded between two negative constants, the geodesic flow ϕ on SM is Anosov. Moreover in this condition, if the volume of M is finite, then the non-wandering set of the geodesic flow $\Omega(\phi^t)$ is equal to SM and the spectral decomposition theorem implies that the geodesic flow is transitive, so $W^{cs}(x, v)$ and $W^{cu}(x, v)$ are dense sets in SM , for any $(x, v) \in SM$ (cf. [Ano69], [Pat99] and [[Kli82] chapter 3]).

The subbundles E^{ss} and E^{uu} are known to be uniquely integrable. They are tangent to the strong stable foliation W^{ss} and strong unstable foliation W^{uu} , (cf. [CL77] for the precise definition of C^r foliation).

3 Hyperbolic Set in Cross-section for Geodesic Flow

In this section we construct a hyperbolic set with Hausdorff dimension greater than 1 for Poincaré map associated with the geodesic flow. First we consider some theorems that will be used in our arguments.

The following theorem was proved by S.G. Dani (cf. [Dan86] and [DV89])

Theorem: *Let M be a complete noncompact Riemannian manifold such that all the sectional curvatures are bounded between two negative constants and the Riemannian volume is finite. Let $p \in M$ and S_p the space of unit tangent vectors at p , let C be the subset of S_p consisting of all elements u such that the geodesic rays starting at p in the direction u is a bounded subset of M . Then C is an incompressible subset of S_p (cf. subsection 5.6).*

In other words, let M be a manifold as in the theorem and consider the geodesic flow corresponding to M , defined on the unit tangent bundle, *i.e.*,

$$SM = \{(p, u) : p \in M, u \in S_p\}$$

equipped with the usual Riemannian metric. Then, the above theorem implies the following result on the dynamics of the flow.

Corollary: *Let the notation be as above and let C be the subset of SM consisting of all elements (p, u) whose orbit under the geodesic flow is a bounded subset of SM , then C is a subset of SM , which has Hausdorff dimension equal to the dimension of SM , with respect to the distance induced by the Riemannian metric.*

In particular, if M is a surface, then $HD(C) = \dim(SM) = 2(2) - 1 = 3$, where HD denotes the Hausdorff Dimension. Using this Corollary we will construct a hyperbolic set for the geodesic flow ϕ .

Consider now a family of bounded open subsets indexed by \mathbb{R} , with the following properties:

1. If $\alpha < \beta$, then $\Omega_\alpha \subset \Omega_\beta$.
2. $\Omega_\alpha \nearrow SM$, this is, $\bigcup_{\alpha \in \mathbb{R}} \Omega_\alpha = SM$.

For example, $\Omega_\alpha = B_\alpha(p)$, the ball of radius α and center p .
Let

$$\begin{aligned} \phi^t : \mathbb{R} \times SM &\longrightarrow SM \\ (t, x) &\longmapsto \phi^t(x) \end{aligned}$$

be the geodesic flow.

Put $\tilde{\Omega}_\alpha = \bigcap_{t \in \mathbb{R}} \phi^t(\Omega_\alpha)$, then we have the following statement:

$$C \subset \bigcup_{\alpha \in \mathbb{R}} \tilde{\Omega}_\alpha,$$

where C is given in the previous Corollary.

In fact, let $x \in C$, then there exists a compact set K_x such that the orbit of x , $O(x) \subset K_x \subset \Omega_{\alpha_x}$ for some $\alpha_x \in \mathbb{R}$, this implies that $\phi^t(x) \in \Omega_{\alpha_x}$ for all $t \in \mathbb{R}$, therefore $x \in \tilde{\Omega}_{\alpha_x}$ and the statement is proved.

Let α_n be a sequence in \mathbb{R} such that $\alpha_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\alpha_n < \alpha_{n+1}$, then $\tilde{\Omega}_{\alpha_n} \subset \tilde{\Omega}_{\alpha_{n+1}}$, since $\Omega_{\alpha_n} \subset \Omega_{\alpha_{n+1}}$. Hence

$$C \subset \bigcup_{n=1}^{\infty} \tilde{\Omega}_{\alpha_n},$$

where $\bar{\Omega}$ is the closure of Ω . Since $HD(C) = 3$, then $\sup_n HD(\tilde{\Omega}_{\alpha_n}) = 3$, therefore there exists n such that $HD(\tilde{\Omega}_{\alpha_n})$ is very close to 3.

Now notice that $\tilde{\Omega}_{\alpha_n}$ is compact and ϕ^t -invariant, and since ϕ^t is an Anosov flow on SM , then $\tilde{\Omega}_{\alpha_n}$ is hyperbolic set for geodesic flow ϕ^t . Call

$$\Lambda := \tilde{\Omega}_{\alpha_n} \text{ and } HD(\Lambda) \sim 3. \tag{2}$$

3.1 Cross-sections and Poincaré Maps

This section is adapted from [AP10, chap. 6].

Let Σ be a cross-section to the flow, that is a C^1 -embedded compact disk transverse a ϕ^t at every point $z \in \Sigma$: We have $T_z \Sigma \oplus \langle \phi(z) \rangle = T_z SM$ (recall that $\langle \phi(z) \rangle$ is the 1-dimensional subspace $\{s\phi(z) : s \in \mathbb{R}\}$). For every $x \in \Sigma$ we define $W^s(x, \Sigma)$ to be the connected component of $W^{cs}(x) \cap \Sigma$ that contains x . This defines a foliation \mathcal{F}_Σ^s of Σ into codimension 1 submanifolds of class C^1 (cf. [AP10]).

Remark 1. Given any cross-section Σ and a point x in its interior, we may always find a smaller cross-section also with x in its interior and which is the image of the square $[0, 1] \times [0, 1]$, by a C^2 diffeomorphism h that sends horizontal lines inside leaves of \mathcal{F}_Σ^s . Thus, the cross section that we consider are those that are image of the square $[0, 1] \times [0, 1]$ by a C^2 diffeomorphism h that sends horizontal lines inside leaves of \mathcal{F}_Σ^s . In this case, we denote by $\text{int}(\Sigma)$ the image of $(0, 1) \times (0, 1)$ under the above-mentioned diffeomorphism, which we call the interior of Σ

3.1.1 Hyperbolicity of Poincaré Maps

Let $\Xi = \bigcup \Sigma_i$ be finite union of cross-sections to the flow ϕ^t and let $\mathcal{R}: \Xi \rightarrow \Xi$ be a Poincaré map or the map of first return to Ξ , $\mathcal{R}(y) = \phi^{t_1(y)}(y)$, where $t_1(y)$ correspond to the first time that the orbits of $y \in \Xi$ encounter Ξ .

The splitting $E^{ss} \oplus \phi \oplus E^{uu}$ over U_0 neighborhood of Λ defines a continuous splitting $E_\Sigma^s \oplus E_\Sigma^u$ of the tangent bundle $T\Sigma$ with $\Sigma \in \{\Sigma_i\}_i$, defined by

$$E_\Sigma^s(y) = E_y^{cs} \cap T_y \Sigma \text{ and } E_\Sigma^u(y) = E_y^{cu} \cap T_y \Sigma \quad (3)$$

where $E_y^{cs} = E_y^{ss} \oplus \langle \phi(y) \rangle$ and $E_y^{cu} = E_y^{uu} \oplus \langle \phi(y) \rangle$.

We now show that for a sufficiently large iterated of \mathcal{R} , \mathcal{R}^n , then (3) define a hyperbolic splitting for transformation \mathcal{R}^n on the cross-sections, at last restricted to Λ .

Remark 2.

1. In what follows we use $K \geq 1$ as a generic notation for large constants depending only on a lower bound for the angles between the cross-sections and the flow direction, and on upper and lower bounds for the norm of the vector field on the cross-sections.

2. Let us consider unit vectors, $e_x^{ss} \in E_x^{ss}$ and $\hat{e}_x^s \in E_\Sigma^s(x)$, and write

$$e_x^{ss} = a_x \hat{e}_x^s + b_x \frac{\phi(x)}{\|\phi(x)\|}. \quad (4)$$

Since the angle between E_x^{ss} and $\phi(x)$, $\angle(E_x^{ss}, \phi(x))$, is greater than or equal to the angle between E_x^{ss} and E_x^{cu} , $\angle(E_x^{ss}, E_x^{cu})$, because $\phi(x) \in E_x^{cu}$ and the latter is uniformly bounded from zero, we have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow. It is clear from (4) and the fact that the above angle is uniformly bounded from zero.

Let $0 < \lambda < 1$ be, then there is $t_1 > 0$ such that $\lambda^{t_1} < \frac{\kappa}{K} \lambda$ and $\lambda^{t_1} < \frac{\lambda}{K^3}$, take n , such that $t_n(x) := \sum_{i=1}^n t_i(x) > t_1$ for all $x \in \Xi$, where $t_i(x)$ is such that $\mathcal{R}^i(x) = \phi^{t_i(x)}(\mathcal{R}^{i-1}(x))$.

So, we have the following proposition:

Proposition 1. Let $\mathcal{R}: \Xi \rightarrow \Xi$ be a Poincaré map and n as before. Then $D\mathcal{R}_x^n(E_\Sigma^s(x)) = E_{\Sigma'}^s(\mathcal{R}^n(x))$ at every $x \in \Sigma \in \{\Sigma_i\}_i$ and $D\mathcal{R}_x^n(E_\Sigma^u(x)) = E_{\Sigma'}^u(\mathcal{R}^n(x))$ at every $x \in \Lambda \cap \Sigma$ where $\mathcal{R}^n(x) \in \Sigma' \in \{\Sigma_i\}_i$.

Moreover, we have that

$$\|D\mathcal{R}^n|_{E_\Sigma^s(x)}\| < \lambda \text{ and } \|D\mathcal{R}^n|_{E_\Sigma^u(x)}\| > \frac{1}{\lambda}$$

at every $x \in \Sigma \in \{\Sigma_i\}_i$.

Proof. The differential of the map \mathcal{R}^n at any point $x \in \Sigma$ is given by

$$D\mathcal{R}^n(x) = P_{\mathcal{R}^n(x)} \circ D\phi^{t_n(x)}|_{T_x\Sigma},$$

where $P_{\mathcal{R}^n(x)}$ is the projection onto $T_{\mathcal{R}^n(x)}\Sigma'$ along the direction of $\phi(\mathcal{R}^n(x))$.

Note that E_Σ^s is tangent to $\Sigma \cap W^{cs} \supset W^s(x, \Sigma)$. Since the center stable manifold $W^{cs}(x)$ is invariant, we have invariance of the stable bundle:

$$D\mathcal{R}^n(x)(E_\Sigma^s(x)) = E_{\Sigma'}^s(\mathcal{R}^n(x)).$$

Moreover, for all $x \in \Sigma$ we have

$$D\phi^{t_n(x)}(E_\Sigma^u(x)) \subset D\phi^{t_n(x)}(E_x^{cu}) = E_{\mathcal{R}^n(x)}^{cu},$$

since $P_{\mathcal{R}^n(x)}$ is the projection along the vector field, it sends $E_{\mathcal{R}^n(x)}^{cu}$ to $E_{\Sigma'}^u(\mathcal{R}^n(x))$.

This proves that the unstable bundle is invariant restricted to Λ , that is, $D\mathcal{R}^n(x)(E_\Sigma^u(x)) = E_{\Sigma'}^u(\mathcal{R}^n(x))$, because has the same dimension 1.

Next, we prove the expansion and contraction statements. We start by noting that $\|P_{\mathcal{R}^n(x)}\| \leq K$, with $K \geq 1$, then we consider the basis $\left\{\frac{\phi(x)}{\|\phi(x)\|}, e_x^u\right\}$ of E_x^{cu} , where e_x^u is a unit vector in the direction of $E_\Sigma^u(x)$ and $\phi(x)$ is the direction of flow. Since the flow direction is invariant, the matrix of $D\phi^t|_{E_x^{cu}}$ relative to this basis is upper triangular:

$$D\phi^{t_n(x)}|_{E_x^{cu}} = \begin{bmatrix} \frac{\|\phi(\mathcal{R}^n(x))\|}{\|\phi(x)\|} & * \\ 0 & a \end{bmatrix}$$

this is due to fact that $D\phi^{t_n(x)}(\phi(x)) = \phi(\phi^{t_n(x)}(x)) = \phi(\mathcal{R}^n(x))$.

Then,

$$\begin{aligned} \|D\mathcal{R}^n(x)e_x^u\| &= \|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))e_x^u\| = \|ae_{\mathcal{R}^n(x)}^u\| = |a| \\ &\geq \frac{1}{K} \frac{\|\phi(x)\|}{\|\phi(\mathcal{R}^n(x))\|} |\det(D\phi^{t_n(x)}|_{E_x^{cu}})| \geq \frac{1}{K^3} \lambda^{-t_n(x)} \geq K^{-3} \lambda^{-t_1} > \frac{1}{\lambda}. \end{aligned}$$

To prove that $\|D\mathcal{R}^n|_{E_\Sigma^s(x)}\| < \lambda$, let us consider unit vectors, $e_x^{ss} \in E_x^{ss}$ and $\hat{e}_x^s \in E_\Sigma^s(x)$, and write as in (4)

$$e_x^{ss} = a_x \hat{e}_x^s + b_x \frac{\phi(x)}{\|\phi(x)\|}.$$

We have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow.

Then, since $P_{\mathcal{R}^n(x)}\left(\frac{\phi(\mathcal{R}^n(x))}{\|\phi(x)\|}\right) = 0$ we have that

$$\begin{aligned}
\|D\mathcal{R}^n(x)\hat{e}_x^s\| &= \|P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))\hat{e}_x^s\| \\
&= \left\| P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x)) \left(\frac{1}{a_x} \left(e_x^{ss} - b_x \frac{\phi(x)}{\|\phi(x)\|} \right) \right) \right\| \\
&= \frac{1}{|a_x|} \left\| P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x)) \left(e_x^{ss} - b_x \frac{\phi(x)}{\|\phi(x)\|} \right) \right\| \\
&= \frac{1}{|a_x|} \left\| P_{\mathcal{R}^n(x)}(D\phi^{t_n(x)}(x))(e_x^{ss}) - b_x P_{\mathcal{R}^n(x)}\left(\frac{\phi(\mathcal{R}^n(x))}{\|\phi(x)\|}\right) \right\| \\
&\leq \frac{K}{\kappa} \|D\phi^{t_n(x)}(x)(e_x^{ss})\| \leq \frac{K}{\kappa} \lambda^{t_n(x)} \leq \frac{K}{\kappa} \lambda^{t_1} < \lambda.
\end{aligned} \tag{5}$$

□

3.2 Good Cross-Sections

For each $x \in \Lambda = \widetilde{\Omega}_{\alpha_n}$, (cf. (2)), we can take cross-section Σ in x , and using a tubular neighborhood construction in the cross-section Σ , we linearize the flow in an open set $U_\Sigma = \phi^{(-\gamma, \gamma)}(\text{int}\Sigma)$ for a small $\gamma > 0$, containing x the interior of the cross section.

This provides an open covering of the compact set Λ by tubular neighborhoods.

We let $\{U_{\Sigma_i} : i = 1, 2, \dots, l\}$ be a finite covering of Λ , this is

$$\Lambda \subset \bigcup_{i=1}^l U_{\Sigma_i} = \bigcup_{i=1}^l \phi^{(-\gamma, \gamma)}(\text{int}\Sigma_i). \tag{6}$$

Using a result on the differentiability of the strong stable foliations, we can choose these cross-sections Σ_i in such a way that they do not intersect.

Now we introduce the tools to prove the above claims.

The following result is due to Morris W. Hirsch & Charles C. Pugh (cf. [HP75]).

Theorem(Smoothness Theorem)

Let M be a complete surface with Gaussian curvature bounded between two negative constants, then the Anosov splitting $T(SM) = E^{ss} \oplus \phi \oplus E^{uu}$ for the geodesic flow is of class C^1 . In particular, the strong stable foliations and strong unstable foliations are of class C^1 .

Let \mathcal{F}^{ss} be the strong stable foliations and \mathcal{F}^{uu} the strong unstable foliations, this is $\mathcal{F}^i(x) = W^i(x)$ for $i = ss, uu$, are foliations of dimension one. Then we have the following Lemma.

Lemma 1. *Let $x \in SM$ and L be a C^1 -embedded curve of dimension one, containing x and transverse to the foliation \mathcal{F}^{ss} , then the set*

$$S_L := \bigcup_{z \in L} \mathcal{F}^{ss}(z)$$

contains a surface S_x that is C^1 -embedded, which contains x in the interior and if L is transverse to the foliation W^{cs} then, S_x is transverse to the geodesic flow.

Proof. Let (U, φ) be a chart of the foliation \mathcal{F}^{ss} , with $x \in U$, since the dimension of foliation \mathcal{F}^{ss} is equal to 1 and $\dim(SM) = 3$, there are disks $U_1 \subset \mathbb{R}$ and $U_2 \subset \mathbb{R}^2$ such that $\varphi : U \rightarrow U_1 \times U_2$, put $\Pi_2 : U_1 \times U_2 \rightarrow U_2$ the projection on the second coordinate. Let $f = \Pi_2 \circ \varphi$ function of class C^1 , clearly f is a submersion.

Claim: If $D\varphi_x(v) = (w, (0, 0)) \in \mathbb{R} \times \mathbb{R}^2$, then $v \in T_x \mathcal{F}^{ss}(x)$.

In fact: Let $\alpha(t)$ be any curve in U , such that $\alpha(0) = x$ and $\alpha'(0) = v$, put $\varphi(x) = (x_1, x_2)$ and $\varphi(\alpha(t)) = (\alpha_1(t), \alpha_2(t))$. Therefore, $\alpha_i(0) = x_i$ for $i = 1, 2$ and $\alpha'_1(0) = w$, $\alpha'_2(0) = (0, 0)$, then $D\varphi_{\varphi(x)}^{-1}(w, (0, 0)) = \frac{d}{dt}\varphi^{-1}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t))|_{t=0}$, where $\tilde{\alpha}_1(t) = x_1 + tw$ and $\tilde{\alpha}_2(t) = x_2$, by the properties of the chart (U, φ) $\varphi^{-1}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t)) = \varphi^{-1}(\tilde{\alpha}_1(t), x_2) \subset \mathcal{F}^{ss}(x)$ and since $\varphi^{-1}(\tilde{\alpha}_1(0), \tilde{\alpha}_2(0)) = \varphi^{-1}(x_1, x_2) = x$, then

$$v = D\varphi_{\varphi(x)}^{-1}(w, (0, 0)) = \frac{d}{dt}\varphi^{-1}(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t))|_{t=0} \in T_x \mathcal{F}^{ss}(x),$$

as wanted.

Now is easy show that the set $f(L)$ is a C^1 -submanifold of dimension one. Indeed, let $\beta : (-\epsilon, \epsilon) \rightarrow SM$ a C^1 -embedding on L in some $y \in L \cap U$, with $\beta(0) = y$. Then as L is transverse to the foliation \mathcal{F}^{ss} and demonstrated above, we have

$$(f \circ \beta)'(t) = D(\Pi_2)_{\varphi(\beta(t))}(D\varphi_{\beta(t)})(\beta'(t)) \neq 0$$

for all t . As L is a C^1 -embedded, then the above implies that $f(L)$ is a C^1 -submanifold of U_2 . Therefore, since f is a submersion and $f(L)$ is a submanifold, then $f^{-1}(f(L))$ is a C^1 -submanifold of SM , with the following property: If $z \in f(L)$, then $f^{-1}(z) = \varphi^{-1}(\Pi_2^{-1}(z)) = \varphi^{-1}(U_1 \times \{z\}) = \mathcal{F}^{ss}(y) \cap U$ where $z = f(y)$ and $y \in L$, and follows the Lemma. □

In particular, taking $L = W_\epsilon^{uu}(x)$ with ϵ given by the stable and unstable manifolds theorem, we call $S_x := \Sigma_x$. Note that an analogous Lemma holds for the foliation \mathcal{F}^{uu} .

Without loss of generality, we can assume that Σ_x is diffeomorphic to the square $[0, 1] \times [0, 1]$. Put $\Sigma_x = \Sigma$, with the horizontal lines $[0, 1] \times \eta$ being mapped to stable sets $W^s(y, \Sigma_x) = W^{ss}(y) \cap \Sigma_x$. The stable-boundary $\partial^s \Sigma$ is the image of $[0, 1] \times \{0, 1\}$, the unstable-boundary $\partial^u \Sigma$ is the image of $\{0, 1\} \times [0, 1]$. Therefore, we have the following definition.

Definition 1. A cross sections is said **δ -Good Cross-Section** for some $\delta > 0$, if satisfies the following:

$$d(\Lambda \cap \Sigma, \partial^u \Sigma) > \delta \text{ and } d(\Lambda \cap \Sigma, \partial^s \Sigma) > \delta$$

where d is the intrinsic distance in Σ , (cf. Figure 1).

A cross-section which is δ -Good Cross-Section for some $\delta > 0$ is said a **Good Cross-Section-GCS**.

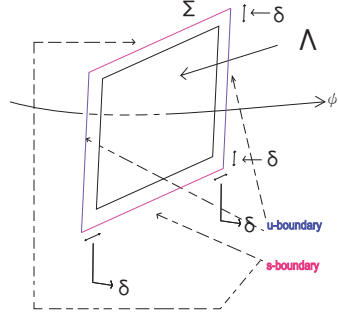


Figure 1: Good Cross-Section

Lemma 2. *Let Σ be a δ -Good Cross-Section, then given $0 < \delta' < \delta$ there is a δ' -Good Cross-Section $\Sigma' \subset \text{int}(\Sigma)$ and such that $\partial\Sigma' \cap \partial\Sigma = \emptyset$.*

Proof. Call γ_i , $i = 1, 2, 3, 4$ the C^1 -curves which form the boundary of Σ . Let γ_i'' be a C^1 -curve contained in Σ and satisfies $d(\gamma_i, x) = \delta$ for any $x \in \gamma_i''$ (cf. Figure 2). Therefore, $\Lambda \cap \Sigma$ is contained in the region bounded by the curves γ_i'' in Σ . Now consider the C^1 -curves $\gamma_i' \subset \Sigma$ with the property $d(\gamma_i, x) = \delta'$ for any $x \in \gamma_i'$. Then the region bounded by the curves γ_i' is a δ' -Good Cross-Section, $\Sigma' \subset \text{int}\Sigma$, (cf. Figure 2).

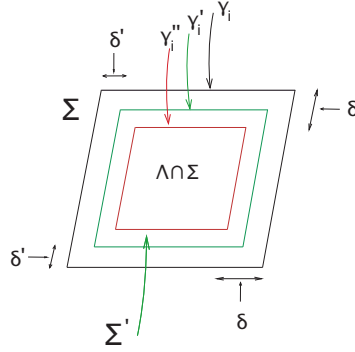


Figure 2: Reduction of GCS

□

Now we prove that for any $x \in \Lambda$ there exists Good Cross-Sections which contains x .

Lemma 3. *For any $x \in \Lambda$ there exist points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in distinct connected components of $W^{ss}(x) - \{x\}$.*

Proof. Let $x \in \Lambda$, suppose otherwise there would exists a whole segment of the strong stable manifold entirely contained in Λ and containing x in the interior, called ζ this segment. Without loss of generality, we can assume that $W_{loc}^{ss}(x) \subset \zeta$. Now take t_k a sequence such that $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Then as Λ is a compact invariant set, we can assume that $\phi^{-t_k}(x) \rightarrow y \in \Lambda$ as $k \rightarrow \infty$.

Claim: $W^{ss}(y) \subset \Lambda$. In fact:

Let $z \in W^{ss}(y)$, as $W^{ss}(y) = \bigcup_{t \geq 0} \phi^{-t}(W_{loc}^{ss}(\phi^t(y)))$, then there is $T \geq 0$, such that $\phi^T(z) \in W_{loc}^{ss}(\phi^T(y))$. Then by Stable Manifold Theorem $W_{loc}^{ss}(\phi^T(y))$ is accumulated by points of $W_{loc}^{ss}(\phi^{(-t_k+T)}(x))$ for large k . Let k be sufficiently large such that $(-t_k+T) < 0$ and $W_{loc}^{ss}(\phi^{(-t_k+T)}(x)) \subset \phi^{(-t_k+T)}(\zeta) \subset \Lambda$, as well Λ is a invariant set and $\zeta \subset \Lambda$. Hence as Λ is closed, we have that $W_{loc}^{ss}(\phi^T(y)) \subset \Lambda$, and this implies that $z \in \Lambda$. This proves the assertion.

The above statement implies that $\Lambda \supset W^{cs}(y) = \bigcup_{t \in \mathbb{R}} W^{ss}(\phi^t(y))$. In fact:

Let $w \in W^{cs}(y)$, then there is $t_0 \in \mathbb{R}$ such that, $w \in W^{ss}(\phi^{t_0}(y))$. Hence there is $T \geq 0$ such that $\phi^T(w) \in W_{\epsilon}^{ss}(\phi^{T+t_0}(y))$. Then $\phi^{T+r}(w) \in W_{K\epsilon e^{-\lambda r}}^{ss}(\phi^{T+r+t_0}(y))$ for $r > 0$, so we can assume that $T + t_0 > 0$. Therefore

$$\phi^{-t_0}(w) = \phi^{-(T+t_0)}(\phi^T(w)) \in \phi^{-(T+t_0)}(W_{\epsilon}^{ss}(\phi^{T+t_0}(y))) \subset W^{ss}(y) \subset \Lambda.$$

Since Λ is invariant, then $w \in \Lambda$. This implies that $W^{cs}(y) \subset \Lambda$, but in our conditions $SM = \overline{W^{cs}(y)} \subset \Lambda$ and this is a contradiction. This concludes the proof of Lemma. \square

Similarly we have,

Lemma 4. *For any $y \in \Lambda$ there are points $y^+ \notin \Lambda$ and $y^- \notin \Lambda$ in distinct connected components of $W^{uu}(x) - \{x\}$.*

Proof. Similar to proof of Lemma 3. \square

Lemma 5. *Let $x \in \Lambda$, then there is $\delta > 0$ and a δ -Good Cross-Section Σ at x .*

Proof. Fix $\epsilon > 0$ as in the Stable Manifold Theorem, and consider the cross section Σ_x given by the Lemma 1 containing a segment of $W_{\epsilon}^{ss}(x)$ and $W_{\epsilon}^{uu}(x)$ with x in the interior. By Lemma 3 and Lemma 4, we may find points $x^{\pm} \notin \Lambda$ in each of the connected components of $W_{\epsilon}^{ss}(x) \cap \Sigma_x$ and points $z^{\pm} \notin \Lambda$ in each of the connected components of $W_{\epsilon}^{uu}(x) \cap \Sigma_x$. Since Λ is closed, there are neighborhoods V^{\pm} of x^{\pm} and V_1^{\pm} of z^{\pm} respectively disjoint from Λ , (cf. Figure 3).

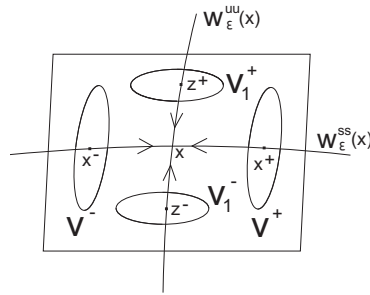


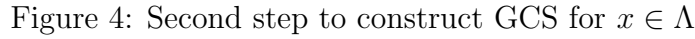
Figure 3: First step to construct GCS for $x \in \Lambda$

In Figure 3, it can happen that V^{\pm} , V_1^{\pm} enclose a region homeomorphic to a square, in this case there is nothing to be done.

If this is not the case in the first instance, we prove that the above can be obtained.

Now for $z \in W_\epsilon^{uu}(x)$, we have

converges to zero as $k \rightarrow \infty$. Using the continuity of $W_\epsilon^{ss}(x)$ with $x \in SM$, given by the Stable Manifold Theorem, we have for sufficiently large k , say $k \geq k_0$, $W_\epsilon^{ss}(\phi^{-t_k}(z))$ is close to $W_\epsilon^{ss}(y)$, for all $z \in W_\epsilon^{uu}(x)$, this implies that $J^\pm \cap W_\epsilon^{ss}(\phi^{-t_k}(z)) \neq \emptyset$. Hence, there are $z_k^\pm \in J^\pm \cap W_\epsilon^{ss}(\phi^{-t_k}(z))$, for all $z \in W_\epsilon^{uu}(x)$ (cf. Figure 4).


$$d(\phi^{t_k}(w_1^+), \phi^{t_k}(w_2^+)) \geq K^{-1}e^{\lambda t_k}d(w_1^+, w_2^+).$$

Note also that as $z_k^+ \in W_\epsilon^{ss}(\phi^{-t_k}(z))$, for $z \in W_\epsilon^{uu}(x)$ then

for $z \in W_\epsilon^{uu}(x)$.

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can obtain k_0 such that $\phi^{t_k}(J^-)$ cross V_1^\pm and is close to $W_\epsilon^{uu}(x)$ for $k \geq k_0$.

On the other hand, we know that for each $z \in W_\epsilon^{uu}(x)$ there is $z_k^\pm \in J^\pm \cap W_\epsilon^{ss}(\phi^{-t_k}(z))$, respectively. Hence, for sufficiently large k_0 , $\phi^{t_{k_0}}(J^+)$ and $\phi^{t_{k_0}}(J^-)$ crossing V^\pm . Moreover, $\phi^{t_{k_0}}(z_{k_0}^\pm) \in W_{Ke^{-\lambda t_{k_0}}}^{ss}(z) \cap \phi^{t_{k_0}}(J^\pm) \subset W_\epsilon^{ss}(z) \cap \phi^{t_{k_0}}(J^\pm) \subset \Sigma_x \cap \phi^{t_{k_0}}(J^\pm)$ for any $z \in W_\epsilon^{uu}(x)$ with $\phi^{t_{k_0}}(J^\pm) \cap \Lambda = \emptyset$. Then the open sets V_1^\pm and $\phi^{t_{k_0}}(J^\pm)$ have the desired property.

Let β^\pm be a segment of $W_\epsilon^{ss}(z^\pm)$ contained in V_1^\pm respectively. Take k_0 large enough such that the endpoints of β^\pm , β_i^\pm for $i = 1, 2$ is contained in $\phi^{t_{k_0}}(J^\pm)$, (cf. Figure 4). Let η^\pm be a C^1 -curve transverse to the foliation \mathcal{F}^{ss} contained in $\phi^{t_{k_0}}(J^\pm) \cap \Sigma_x$ and joining β_1^\pm with β_2^\pm , respectively. Finally, the good cross-section it is the cross-section determined by the curves β^\pm and η^\pm , (cf. Figure 5).

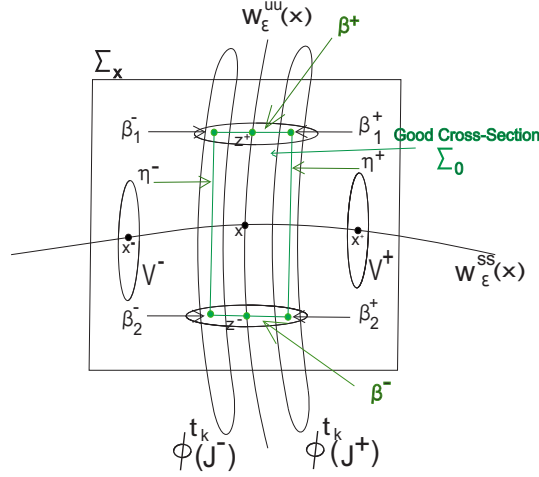


Figure 5: The construction of GCS for $x \in \Lambda$ using positive iterated

And this concludes the proof of the Lemma. □

Remark 3. Note that if $k \geq k_0$, is as in the proof of the Lemma 5, this is, we have the Figure 5, then for $k' \geq k \geq k_0$. We have the same Figure 5, but the open $\phi^{t_{k'}}(J^\pm)$, has diameter much greater than ϵ , (cf. Figure 6).

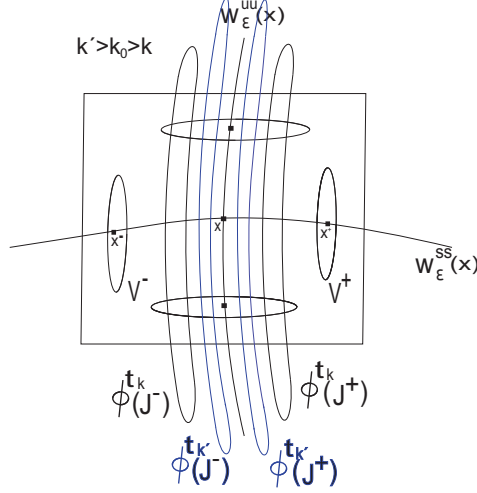


Figure 6: Small GCS

Remark 4. In the proof of Lemma 5, we could consider an accumulation point $\phi^t(x)$ for $t > 0$, and get the same result. But in this case crossed V^\pm , consequently satisfies Remark 3 in this case, (cf. Figure 7).

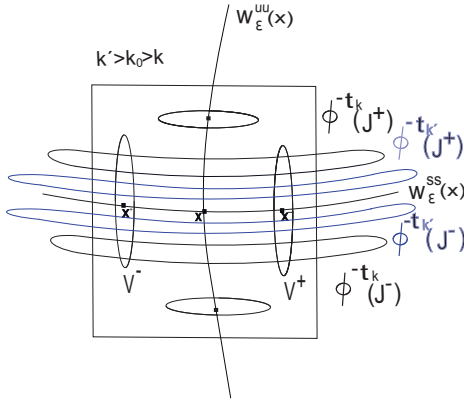


Figure 7: The construction of GCS for $x \in \Lambda$ using negative iterated

Remark 5. Given $x \in \Lambda$ (cf. (2)) from now on, we call Σ_x the Good Cross-Section given by the previous Lemma associated to x .

Corollary 1. Given $x, y \in \Lambda$, such that there is a C^1 -curve $\zeta \subset \text{int}(\Sigma_x) \cap \text{int}(\Sigma_y)$. If ζ intersects transversely to foliation \mathcal{F}^{ss} , then $\text{int}(\Sigma_x) \cap \text{int}(\Sigma_y)$ is an open set of Σ_x and Σ_y .

Proof. Since $\zeta \subset \text{int}(\Sigma_x) \cap \text{int}(\Sigma_y)$ a C^1 -curve transverse to \mathcal{F}^{ss} . Then for all $z \in \zeta$, there are $x' \in W_\epsilon^{uu}(x)$ and $y' \in W_\epsilon^{uu}(y)$ such that $z \in W^{ss}(x') \cap \Sigma_x$ and $z \in W^{ss}(y') \cap \Sigma_y$. Then there is $\delta > 0$ such that the set

$$B = \bigcup_{z \in \gamma} W_\delta^{ss}(z) \subset \text{int}(\Sigma_x) \cap \text{int}(\Sigma_y).$$

Thus, we have the Corollary. \square

Remark 6. Suppose that Σ_1, Σ_2 are GCS and $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, but $\text{int}\Sigma_1 \cap \text{int}\Sigma_2 = \emptyset$, then as both are GCS, there are two GCS $\tilde{\Sigma}_i \subset \Sigma_i$ for $i = 1, 2$ such that $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2 = \emptyset$ with

$$\Lambda \cap \bigcup_{i=1}^2 \phi^{(-\gamma, \gamma)}(\text{int}\Sigma_i) = \Lambda \cap \bigcup_{i=1}^2 \phi^{(-\gamma, \gamma)}(\text{int}\tilde{\Sigma}_i). \quad (7)$$

In fact:

By Lemma 2, there are GCS, $\tilde{\Sigma}_i \subset \text{int}(\Sigma_i)$ such that $\partial\tilde{\Sigma}_i \cap \partial\Sigma_i = \emptyset$ for $i = 1, 2$, as $\text{int}\Sigma_1 \cap \text{int}\Sigma_2 = \emptyset$, then $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2 = \emptyset$. Also the Lemma 2 implies that $\Lambda \cap \text{int}(\Sigma_i) \subset \text{int}(\tilde{\Sigma}_i)$, thus we have $\Lambda \cap \text{int}(\Sigma_i) = \Lambda \cap \text{int}(\tilde{\Sigma}_i)$ and as Λ is ϕ^t invariant. Then

$$\Lambda \cap \phi^{(-\gamma, \gamma)}(\text{int}(\Sigma_i)) = \phi^{(-\gamma, \gamma)}(\Lambda \cap \text{int}(\Sigma_i)) = \phi^{(-\gamma, \gamma)}(\Lambda \cap \text{int}(\tilde{\Sigma}_i)) = \Lambda \cap \phi^{(-\gamma, \gamma)}(\text{int}(\tilde{\Sigma}_i)).$$

Therefore we have (7).

Thus, by Remark 7, from now on we can assume that if two GCS has nonempty intersection, then their interiors have nonempty intersection.

3.3 Separation of GCS

By Lemma 5, at each point of $x \in \Lambda$, we can find a Good Cross-Section Σ_x . Since Λ is a compact set, then as in (6), there are a finite number of points $x_i \in \Lambda$, $i = 1, \dots, l$ such that

$$\Lambda \subset \bigcup_{i=1}^l \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}\Sigma_i) \subset \bigcup_{i=1}^l \phi^{(-\gamma, \gamma)}(\text{int}\Sigma_i) = \bigcup_{i=1}^l U_{\Sigma_i}, \quad (8)$$

where $\Sigma_i := \Sigma_{x_i}$.

In this section we prove that the $\{\Sigma_i\}_i$ can be taken pairwise disjoint.

Lemma 6. If $\Sigma_i \cap \Sigma_j \neq \emptyset$ for some $i, j \in \{1, \dots, l\}$, Σ_i and Σ_j as in the Corollary 1. Then there is $\delta' > 0$ such that $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$ for all $0 < \delta \leq \delta'$.

Proof. Suppose otherwise, then for all n sufficiently large, there is $z_i^n \in \Sigma_i$ such that $\phi^{\frac{1}{n}}(z_i^n) \in \Sigma_j$. Since Σ_i is a compact set, we can assume that z_i^n converge to z_i as n tends to infinity. Then $\phi^{\frac{1}{n}}(z_i^n)$ converge to z_i as n tends to infinity. This implies that $z_i \in \Sigma_i \cap \Sigma_j$.

Suppose that $z_i \in \text{int}\Sigma_j$, as the vector field which generates the geodesic flow has no singularities, then by the Tubular Flow Theorem, there are $r > 0$ and $\eta > 0$ such that $B_r(z_i)$, the open ball of radius r and center z_i , satisfies

$$\phi^t(B_r(z_i) \cap \Sigma_j) \cap \Sigma_j = \emptyset$$

for all $0 < t \leq \eta$.

Moreover, by the Corollary 1, we have $(B_r(z_i) \cap \Sigma_i) \setminus \{z_i\} \subset \Sigma_j$. Take n large enough such that $z_i^n \in B_r(z_i) \cap \Sigma_i$ and $\frac{1}{n} < \eta$. So $\phi^{\frac{1}{n}}(z_i^n) \notin \Sigma_j$ which is a contradiction.

Suppose now that $z_i \in \partial \Sigma_j$. Then we can find a new GCS $\Sigma'_j \supset \Sigma_j$ as in the Lemma 5 and such that $z_i \in \text{int} \Sigma'_j$. So Σ_i and Σ'_j behave as in the previous case and again to obtain a contradiction. Thus we conclude the Lemma. \square

The following Lemma proves that the GCS in (8) can be taken disjoint if all possible intersections of Σ_i with Σ_j are in the hypothesis of the Corollary 1.

Lemma 7. *Assuming (8) there are GCS $\tilde{\Sigma}_i$ such that $\Lambda \subset \bigcup_{i=1}^l \phi^{(-\gamma, \gamma)}(\tilde{\Sigma}_i)$ with the property $\tilde{\Sigma}_i \cap \tilde{\Sigma}_j = \emptyset$ for all $i, j \in \{1, \dots, l\}$.*

Proof. We will do the proof by induction on l . If $l = 1$, is clearly true. For $l = 2$, can happen two cases:

1. $\Sigma_1 \cap \Sigma_2 = \emptyset$, in this case take $\tilde{\Sigma}_k = \Sigma_k$ for $k = 1, 2$.
2. $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ and $\text{int} \Sigma_1 \cap \text{int} \Sigma_2 = \emptyset$, then by Remark 6 and (7) have the desired.
3. $\text{int}(\Sigma_1) \cap \text{int}(\Sigma_2) \neq \emptyset$, then take $\delta' > 0$ as in Lemma 6, then for $0 < \delta < \delta'$ and $\delta < \frac{\gamma}{2}$, by (8) putting $\tilde{\Sigma}_1 = \phi^\delta(\Sigma_1)$, clearly $\tilde{\Sigma}_1$ is a GCS and as $\delta < \frac{\gamma}{2}$, then $\phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\Sigma_1) \subset \phi^{(-\gamma, \gamma)}(\tilde{\Sigma}_1)$ and satisfies the Lemma.

Suppose that the Lemma is true for all $k < l$, and we will show that it holds for $k = l$. In fact: Suppose that given any number $k < l$ of GCS as in (8) there are a number $k < l$ of new GCS such that

$$\bigcup_{s=1}^k \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\Sigma_{i_s}) \subset \bigcup_{s=1}^k \phi^{(-\gamma, \gamma)}(\tilde{\Sigma}_{i_s}) \quad (9)$$

and $\tilde{\Sigma}_{i_s} \cap \tilde{\Sigma}_{i_r} = \emptyset$ for $s, r \in \{1, \dots, k\}$.

Note also by Remark 6, we can suppose that, $\Sigma_i \cap \Sigma_j = \emptyset \Leftrightarrow d(\Sigma_i, \Sigma_j) := \delta_{ij} > 0$, where d is the distance between the two cross-section.

Statements:

1. If $\Sigma_i \cap \Sigma_j = \emptyset$ and $\Sigma_k \cap \Sigma_i \neq \emptyset$, then there is $\delta > 0$ such that $\phi^\delta(\Sigma_i) \cap \Sigma_k = \emptyset$ and $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$. In fact:

Let Σ_k be such that $\Sigma_i \cap \Sigma_k \neq \emptyset$, then take $\delta < \min\{\delta_{ij}, \frac{\epsilon}{2}\}$ in Lemma 6 such that $\phi^\delta(\Sigma_i) \cap \Sigma_k = \emptyset$. Moreover, if $z \in \phi^\delta(\Sigma_i) \cap \Sigma_j$, then $\phi^{-\delta}(z) \in \Sigma_i$ and $d(\phi^{-\delta}(z), z) = \delta < \delta_{ij} = d(\Sigma_i, \Sigma_j)$, which is absurd. Therefore $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$.

2. Given $i \in \{1, \dots, l\}$, call $B_i = \{j : \Sigma_i \cap \Sigma_j \neq \emptyset\}$. Then there is $\delta > 0$ such that $\phi^\delta(\Sigma_i) \cap \Sigma_j = \emptyset$ for all j . In fact:

If $r \notin B_i$, then $d(\Sigma_i, \Sigma_r) = \delta_{ir} > 0$. By Lemma 6 for each $s \in B_i$, there is $\delta_s < \min\{\min_{r \notin B_i} \delta_{ir}, \frac{\gamma}{2}\}$ such that $\phi^{\delta_s}(\Sigma_i) \cap \Sigma_s = \emptyset$ and by the choice of δ_s , we also have to $\phi^{\delta_s}(\Sigma_i) \cap \Sigma_r = \emptyset$ for any $r \notin B_i$. So for $\delta = \min_{s \in B_j} \delta_s$ we have the statements.

Now fix the GCS Σ_1 . Suppose that $\#\{j : \Sigma_j \cap \Sigma_1 \neq \emptyset\} := C_1 \leq l - 1$, then from statements 2 above, there is δ such that $\phi^\delta(\Sigma_1) \cap \Sigma_j = \emptyset$ for all $j \neq 1$. Then by induction hypothesis applied to $\{\Sigma_j : j \neq 1\}$, we obtain new GCS $\tilde{\Sigma}_j$ than satisfies (9) and calling $\phi^\delta(\Sigma_1) = \tilde{\Sigma}_1$. Then the set $\{\tilde{\Sigma}_j : j = 1, \dots, l\}$ satisfies the Lemma.

Note that since $d(\tilde{\Sigma}_1, \Sigma_j) = \delta_{1j} > 0$ for all $j \neq 1$, then $\tilde{\Sigma}_j$ may be obtained such that $d(\tilde{\Sigma}_1, \tilde{\Sigma}_j) > 0$.

Suppose now that $\#C_1 = l$, then for each $j \neq 1$ there is $\delta_j > 0$ given by the Lemma 6, such that $\phi^t(\Sigma_1) \cap \Sigma_j = \emptyset$ for all $0 < t \leq \delta_j$. Take $0 < \delta < \min_{j \neq 1} \{\delta_j, \frac{\gamma}{2}\}$. Therefore $\tilde{\Sigma}_1 := \phi^\delta(\Sigma_1)$, satisfies $\tilde{\Sigma}_1 \cap \Sigma_j = \emptyset$ for all $j \neq 1$. Considering $\{\tilde{\Sigma}_1, \Sigma_2, \dots, \Sigma_l\}$, we have $\#\{j : \Sigma_j \cap \Sigma_2 \neq \emptyset\} \leq l - 1$, as done previously, we have the result of Lemma. \square

Now we will study the case of transverse intersections between the sections Σ_i .

Let Σ, Σ' are GCS as in the Lemma 5 with $\Sigma \cap \Sigma' \neq \emptyset$. Suppose that $\Sigma \cap \Sigma'$ is non-transverse to \mathcal{F}^{ss} . Then since Σ, Σ' are transverse to flow, then we can assume that Σ and Σ' intersect transversely.

Suppose now that two GCS Σ_x, Σ' as in the Lemma 5 intersect transversely. Then $\Sigma_x \pitchfork \Sigma'$ is a finite number of C^1 -curve γ_i for $i = 1, \dots, k$ and by Corollary 1 these curves are contained in a finite number of leaves of $\mathcal{F}^{ss} \cap \Sigma_x$, say $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ with $z_i \in \mathcal{F}^{uu}(x) \cap \Sigma_x$ for $i = 1, \dots, k$.

Let $\bar{\Sigma}_i$ be surface contained in Σ_x , containing $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ and saturate by \mathcal{F}^{ss} , *i.e.*, there is an interval I_i contained in $\mathcal{F}^{uu}(x)$ and centered in z_i such that

$$\bar{\Sigma}_i = \bigcup_{z \in I_i} \mathcal{F}^{ss}(z) \cap \Sigma_x \text{ for } i = 1, \dots, k.$$

Since $z_i \neq z_j$, then we can assume that $\bar{\Sigma}_i \cap \bar{\Sigma}_j = \emptyset$ for $i \neq j$.

Note that if $\mathcal{F}^{ss}(z_i) \cap \Sigma_x \cap \Lambda = \emptyset$ for some i , then since Λ is a compact set there is an open set U_i containing $\mathcal{F}^{ss}(z_i) \cap \Sigma_x$ and $U_i \cap \Lambda = \emptyset$. Therefore, Σ_x is subdivided into two GCS Σ_x^1 and Σ_x^2 , such that Σ_x^r and Σ' intersect transversely for $r = 1, 2$.

The above implies that, without loss of generality, we can assume that for any $i \in \{1, \dots, k\}$ there is $p_i \in \mathcal{F}^{ss}(z_i) \cap \Sigma_x \cap \Lambda$, (cf. Figure 8).

Remark 7. Let Σ'' be a GCS as in the Lemma 5 such that $\Sigma_x \cap \Sigma'' = \emptyset$. Taking $\delta < d(\Sigma_x, \Sigma'')$, we have $\phi^\delta(\tilde{\Sigma}_i) \cap \Sigma'' = \emptyset$ for $i \in \{1, \dots, k\}$, where $\tilde{\Sigma}_i$ as in Lemma 8.

Now remember (8)

$$\Lambda \subset \bigcup_{i=1}^l \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int} \Sigma_i) \subset \bigcup_{i=1}^l \phi^{(-\gamma, \gamma)}(\text{int} \Sigma_i) = \bigcup_{i=1}^l U_{\Sigma_i}.$$

In Lemma 7 was proved that the GCS in (8) can be taken disjoint if all possible intersections of Σ_i with Σ_j are in the hypothesis of the Corollary 1.

Now we will prove that the GCS in (8) can be taken disjoint, even if some of them intersect transversely.

Lemma 9. Let Σ_i be a GCS as in (8). Let $B_i = \{j : \Sigma_i \pitchfork \Sigma_j\}$. Then, Σ_i can be subdivided in a finite number of GCS $\{\Sigma_i^s : s = 1, \dots, m\}$ and for each s there is $0 < \delta_s < \frac{\gamma}{2}$ such that

1. $\phi^{\delta_s}(\Sigma_i^s) \cap \Sigma_j = \emptyset$ for any $j \in B_i$ and $\phi^{\delta_s}(\Sigma_i^s) \cap \phi^{\delta_{s'}}(\Sigma_i^{s'}) = \emptyset$ for $s \neq s'$.

2. $\Lambda \cap \bigcup_{j \in B_i \cup \{i\}} \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int} \Sigma_j) \subset \Lambda \cap \left(\bigcup_{j \in B_i} \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_j)) \cup \bigcup_{s=1}^m \phi^{(-\gamma, \gamma)}(\text{int}(\phi^{\delta_s}(\Sigma_i^s))) \right).$

Proof. The proof is by induction on $\#B_i$. The case $\#B_i = 1$ is true by the Lemma 8. Suppose the statement is true for $\#B_i < q$ and we prove for $\#B_i = q$. In fact:

Let $k \in B_i$, then by Lemma 8, given $0 < \delta < \frac{\gamma}{2}$, there are a finite number of GCS $\{\tilde{\Sigma}_k^r \subset \Sigma_k : r \in \{1, \dots, r_k\}\}$ such that

$$\phi^\delta(\tilde{\Sigma}_k^r) \cap \Sigma_k = \emptyset, \text{ also } \phi^\delta(\tilde{\Sigma}_k^r) \cap \Sigma_i = \emptyset \text{ for any } r, \Sigma_i \cap \Sigma = \emptyset \text{ for any } \Sigma \in \Sigma_k^\#. \quad (10)$$

$$\Lambda \cap \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_k)) \subset \Lambda \cap \left(\bigcup_{r=1}^{r_k} \phi^{(-\gamma, \gamma)}(\phi^\delta(\text{int}(\tilde{\Sigma}_k^r))) \cup \bigcup_{\Sigma \in \Sigma_k^\#} \phi^{(-\gamma, \gamma)}(\text{int}(\Sigma)) \right) \quad (11)$$

$\Sigma_k^\#$ as in Lemma 8.

Consider now the set of GCS $\Sigma_i \cup \{\Sigma_j : j \in B_i \setminus \{k\}\} \cup \{\phi^\delta(\tilde{\Sigma}_k^r) : r \in \{1, \dots, r_k\}\} \cup \Sigma_k^\#$. For this new set of GCS, we have $\#B_i < q$. Therefore, by the induction hypothesis, the Lemma is true for $\{\Sigma_j : j \in B_i \cup \{i\} \setminus \{k\}\}$ and by (10) and (11) we have the Lemma. \square

Remark 8. Let Σ_p be a GCS as in (8) such that $\Sigma_p \cap \Sigma_i = \emptyset$. Then by Remark 7, δ_s can be taken less than $d(\Sigma_i, \Sigma_p)$. So $\phi^{\delta_s}(\Sigma_i^s) \cap \Sigma_p = \emptyset$ for any $s \in \{1, \dots, m\}$, Σ_i^s as in Lemma 9.

Let Σ_i be GCS as in (8) where $\Lambda \subset \bigcup_{i=1}^l \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_i))$. We can assume that the possible intersections of $\text{int}(\Sigma_i)$ with $\text{int}(\Sigma_j)$ are as in Corollary 1 or transverse. Then,

Lemma 10. *There are GCS $\tilde{\Sigma}_i$ such that*

$$\Lambda \subset \bigcup_{i=1}^{m(l)} \phi^{(-2\gamma, 2\gamma)}(\text{int}(\tilde{\Sigma}_i))$$

with $\tilde{\Sigma}_i \cap \tilde{\Sigma}_j = \emptyset$.

Proof. If all the possible intersections are as in Corollary 1, the result follows from Lemma 7. Then, we can suppose that there is i , such that $B_i = \{j : \Sigma_i \pitchfork \Sigma_j\} \neq \emptyset$. Without loss of generality, assume that $B_1 \neq \emptyset$. The proof is by induction on l in (8).

The Lemma 8 implies the case $l = 2$. Suppose true for $k < l$ and we prove for $k = l$. In fact:

Fix Σ_1 , call $T_1 = \{j : \Sigma_j \text{ intersect } \Sigma_1 \text{ as in Corollary 1}\}$. Then by statement 2 in the proof of Lemma 7, there is $0 < \delta < \frac{\gamma}{4}$ small, such that $\phi^\delta(\Sigma_1) \cap \Sigma_j = \emptyset$ for any $j \in T_1$.

Consider now the GCS $\phi^\delta(\Sigma_1)$ as in Lemma 5. Let still call $B_1 = \{j : \phi^\delta(\Sigma_1) \pitchfork \Sigma_j\}$. Then by Lemma 9, $\phi^\delta(\Sigma_1)$ can be subdivided in a finite number of GCS $\{\Sigma_1^s : s = 1, \dots, m\}$ and for each s there is $0 < \delta_s < \frac{\gamma}{2}$ such that holds 1 and 2 of Lemma 9. Also by Remark 8 we can assume that $\phi^{\delta_s}(\Sigma_1^s) \cap \Sigma_j = \emptyset$ for any $s \in \{1, \dots, m\}$ and any $j \in T_1 \setminus \{1\}$.

Now take the set $\{\Sigma_j : j \in T_1 \setminus \{1\}\} \cup \{\Sigma_k : k \in B_1\}$, as $\#(T_1 \setminus \{1\} \cup B_1) < l$, then by the induction hypothesis there are GCS $\tilde{\Sigma}_i$, $\tilde{\Sigma}_i \cap \tilde{\Sigma}_j = \emptyset$ for $i \neq j$, such that

$$\Lambda \cap \bigcup_{i \in T_1 \cup B_1 \setminus \{1\}} \left(\phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_i)) \right) \subset \Lambda \cap \bigcup_{i=2}^{n(l)} \phi^{(-\gamma, \gamma)}(\text{int}(\tilde{\Sigma}_i)). \quad (12)$$

Since $\phi^{\delta_s}(\Sigma_1^s) \cap \Sigma_j = \emptyset$ for any $j \in T_1 \cup B_1 \setminus \{1\}$ and any $s \in \{1, \dots, m\}$, then the $\tilde{\Sigma}_j$ may be taken such that $\phi^{\delta_s}(\Sigma_1^s) \cap \tilde{\Sigma}_i = \emptyset$ for any $s \in \{1, \dots, m\}$ and any $i \in \{2, \dots, n(l)\}$.

So, by 2 of Lemma 9 and (12) we have that

$$\begin{aligned} \Lambda &= \Lambda \cap \bigcup_{j=1}^l \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_j)) \subset \Lambda \cap \left(\bigcup_{j=2}^l \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_j)) \cup \phi^{(-\gamma, \gamma)}(\text{int}(\phi^\delta(\Sigma_1))) \right) = \\ &= \Lambda \cap \left(\bigcup_{j \in B_1} \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_j)) \cup \bigcup_{j \in T_1 \setminus \{1\}} \phi^{(-\frac{\gamma}{2}, \frac{\gamma}{2})}(\text{int}(\Sigma_j)) \cup \phi^{(-\gamma, \gamma)}(\text{int}(\phi^\delta(\Sigma_1))) \right) \\ &\subset \Lambda \cap \left(\bigcup_{i=2}^{n(l)} \phi^{(-\epsilon, \epsilon)}(\text{int}(\tilde{\Sigma}_i)) \cup \bigcup_{s=1}^k \phi^{(-2\gamma, 2\gamma)}(\text{int}(\phi^{\delta_s}(\Sigma_1^s))) \right). \end{aligned}$$

This concludes our proof. □

3.4 Global Poincaré Map

Let $\mathcal{R}: \Xi \rightarrow \Xi$ be a Poincaré map as in the section 3.1.1 with $\Xi = \{\Sigma_1, \dots, \Sigma_l\}$, where Σ_i are GCS and $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. Since the GCS $\{\Sigma_i\}_i$ are pairwise disjoint, then without loss of generality, sometimes we denote also $\Xi = \bigcup_{i=1}^l \Sigma_i$.

We will show that, if the cross-section of Ξ are δ -Good Cross-Section (GCS), we have the invariance property

$$\mathcal{R}^n(W^s(x, \Sigma)) \subset W^s(\mathcal{R}^n(x), \Sigma'),$$

for some n sufficiently large.

Given $\Sigma, \Sigma' \in \Xi$ we set $\Sigma(\Sigma')_n = \{x \in \Sigma : \mathcal{R}^n(x) \in \Sigma'\}$ the domain of the map \mathcal{R}^n from Σ to Σ' . Remember relation (5) from the proof of proposition 1, the tangent direction to each $W^s(x, \Sigma)$ is contracted at an exponential rate $\|D\mathcal{R}^n(x)\hat{e}_x^s\| \leq Ce^{-\beta t_n(x)}$, with $C = \frac{K}{\kappa}$ and $\beta = -\log \lambda > 0$. Suppose that the cross-sections in Ξ are δ -GCS. Take n such that $t_n(x) > t_1$ as in proposition 1 with t_1 satisfying

$$Ce^{-\beta t_1} \sup \{l(W^s(x, \Sigma)) : x \in \Sigma\} < \delta \text{ and } Ce^{-\beta t_1} < \frac{1}{2}, \quad (13)$$

where $l(W^s(x, \Sigma))$ is the length of $W^s(x, \Sigma)$.

Lemma 11. *Let n be satisfying the above. Given δ -Good Cross-Sections, $\Sigma, \Sigma' \in \{\Sigma_i\}_i$. If $\mathcal{R}^n: \Sigma(\Sigma')_n \rightarrow \Sigma'$ defined by $\mathcal{R}^n(z) = \phi^{t_n(z)}(z)$. Then,*

1. $\mathcal{R}^n(W^s(x, \Sigma)) \subset W^s(\mathcal{R}^n(x), \Sigma')$ for every $x \in \Sigma(\Sigma')_n$, and also
2. $d(\mathcal{R}^n(y), \mathcal{R}^n(z)) \leq \frac{1}{2}d(y, z)$ for every $y, z \in W^s(x, \Sigma)$ and $x \in \Sigma(\Sigma')_n$.

We let $\{U_{\Sigma_i} : i = 1, \dots, l\}$ be a finite cover of Λ , as in the Lemma 10 where Σ_i are GCS, and we set T_3 to be an upper bound for the time it takes any point $z \in U_{\Sigma_i}$ to leave this tubular neighborhood under the flow, for any $i = 1, \dots, l$. We assume without loss of generality that $t_1 > T_3$.

Let t_1 be as in equation (13) and consider \mathcal{R}^n . If the point z never returns to one of the cross-sections, then the map \mathcal{R} is not defined at z . Moreover, by the Lemma 11, if \mathcal{R}^n is defined for $x \in \Sigma$ on some $\Sigma \in \Xi$, then \mathcal{R} is defined for every point in $W^s(x, \Sigma)$. Hence, the domain of $\mathcal{R}^n|_{\Sigma}$ consists of strips of Σ . The smoothness of $(t, x) \rightarrow \phi^t(x)$ ensure that the strips

$$\Sigma(\Sigma')_n = \{x \in \Sigma : \mathcal{R}^n(x) \in \Sigma'\}$$

have non-empty interior in Σ for every $\Sigma, \Sigma' \in \Xi$. Note that \mathcal{R} is locally smooth for all points $x \in \text{int}\Sigma$ such that $\mathcal{R}(x) \in \text{int}(\Xi)$, by the Tubular Flow Theorem and the smoothness of the flow, where $\text{int}(\Xi) = \{\text{int}\Sigma_i\}_{i=1}^l$. Denote $\partial^j \Xi = \{\partial^j \Sigma_i\}_{i=1}^l$ for $j = s, u$.

Lemma 12. *The set of discontinuities of \mathcal{R} in $\Xi \setminus (\partial^s \Xi \cup \partial^u \Xi)$ is contained in the set of point $x \in \Xi \setminus (\partial^s \Xi \cup \partial^u \Xi)$ such that, $\mathcal{R}(x)$ is defined and belongs to $(\partial^s \Xi \cup \partial^u \Xi)$.*

Proof. Let x be a point in $\Sigma \setminus (\partial^s \Sigma \cup \partial^u \Sigma)$ for some $\Sigma \in \Xi$, not satisfying the condition. Then $\mathcal{R}(x)$ is defined and $\mathcal{R}(x)$ belongs to the interior of some cross-section Σ' . By the smoothness of the flow we have that \mathcal{R} is smooth in a neighborhood of x in Σ . Hence, any discontinuity point for \mathcal{R} must be in the condition of the Lemma. \square

Let $D_j \subset \Sigma_j$ be the set of points sent by \mathcal{R}^n into stable boundary points of some Good Cross-Section of Ξ , if we define the set

$$L_j = \{W^s(x, \Sigma_j) : x \in D_j\},$$

then the Lemma 11 implies that $L_j = D_j$. Let $B_j \subset \Sigma_j$ be the set of points sent by \mathcal{R}^n into unstable boundary points of some Good Cross-Section of Ξ . Denote

$$\Gamma_j = \bigcup_{x \in D_j} W^s(x, \Sigma_j) \cup B_j \quad \text{and} \quad \Gamma = \bigcup \Gamma_j \cup (\partial^s \Xi \cup \partial^u \Xi).$$

Then, \mathcal{R}^n is smooth in the complement $\Xi \setminus \Gamma$ of Γ . Observe that if $x \in D_j$ for some $j \in \{1, \dots, l\}$, then

$$\mathcal{R}^n(W^s(x, \Sigma_j)) \subset \partial^s \Sigma' \quad \text{for some } \Sigma' \in \Xi.$$

We know that $\partial^s \Xi \cap \Lambda = \emptyset$, then $\mathcal{R}^n(W^s(x, \Sigma_j)) \cap \Lambda = \emptyset$. This implies that $W^s(x, \Sigma_j) \cap \Lambda = \emptyset$ for all $x \in D_j$. Moreover, if $x \in B_j$, then $\mathcal{R}^n(x) \in \partial^u \Sigma'$ for some $\Sigma' \in \Xi$, we know that $\partial^u \Xi \cap \Lambda = \emptyset$, this implies that $B_j \cap \Lambda = \emptyset$. Therefore, $\Gamma_j \cap \Lambda = \emptyset$ for all $j \in \{1, \dots, l\}$, so $\Gamma \cap \Lambda = \emptyset$. Thus, if $x \in \Lambda \cap \Xi$, then $\mathcal{R}^n(x)$ is defined and $\mathcal{R}^n(x) \in \text{int}(\Xi)$.

Let $x \in \Lambda \cap \Sigma_j$, then $x \in \Sigma_j \setminus (\Gamma_j \cup \partial^s \Sigma_j \cup \partial^u \Sigma_j)$ and $\mathcal{R}^n(x)$ is defined and $\mathcal{R}^n(x) \in \Sigma_i \cap \Lambda$ for some $i \in \{1, \dots, l\}$. The above implies that $\Lambda \cap \Xi$ is an invariant set for \mathcal{R}^n and by Proposition 1, $\Lambda \cap \Xi$ is hyperbolic set for \mathcal{R}^n and since $\Lambda \cap \Xi$ is invariant for \mathcal{R} , then $\Lambda \cap \Xi$ is hyperbolic for \mathcal{R} , and

$$\Lambda \cap \Xi \subset \bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} \left(\bigcup_{i=1}^l \Sigma_i \right).$$

3.5 Hausdorff Dimension of Hyperbolic set of \mathcal{R}

Now we are going to estimate of $HD(\Lambda \cap \bigcup_{i=1}^l \Sigma_i)$.

Lemma 13. *The set Λ satisfies*

$$\Lambda \subset \bigcup_{t \in \mathbb{R}} \phi^t \left(\bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} (\Lambda \cap \bigcup_{i=1}^l \Sigma_i) \right) = \bigcup_{t \in \mathbb{R}} \phi^t (\Lambda \cap \bigcup_{i=1}^l \Sigma_i) \subset \bigcup_{t \in \mathbb{R}} \phi^t \left(\bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} \left(\bigcup_{i=1}^l \Sigma_i \right) \right).$$

Proof. Remember that $\Lambda \subset \bigcup_{i=1}^l U_{\Sigma_i}$, where $U_i = \phi^{(-\epsilon, \epsilon)}(\text{int} \Sigma_i)$. Let $z \in \Lambda$, then there is t_z such that $z = \phi^{t_z}(x)$ with $x \in \text{int}(\Sigma_s)$ for some s . This implies that $x \in \Lambda \cap \bigcup_{i=1}^l \Sigma_i$ and $\mathcal{R}(x) \in \text{int}(\Sigma_j)$ for some j , so $\mathcal{R}(x) \in \text{int}(\Xi)$. Analogously, $\mathcal{R}^n(x) \in \text{int}(\Xi)$, i.e., $\mathcal{R}^n(x) \in \Lambda \cap \Xi$ for all $n \in \mathbb{Z}$. Hence, $x \in \bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} (\Lambda \cap (\bigcup_{i=1}^l \Sigma_i))$, therefore $z \in \phi^{t_z} \left(\bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} (\Lambda \cap (\bigcup_{i=1}^l \Sigma_i)) \right)$. \square

Lemma 14. *The Hausdorff Dimension of $\Lambda \cap (\bigcup_{i=1}^l \Sigma_i)$ and $\bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} (\bigcup_{i=1}^l \Sigma_i)$ satisfies,*

$$HD \left(\bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n} \left(\bigcup_{i=1}^l \Sigma_i \right) \right) \geq HD \left(\Lambda \cap \left(\bigcup_{i=1}^l \Sigma_i \right) \right) \geq HD(\Lambda) - 1$$

and thus $HD(\Lambda \cap (\bigcup_{i=1}^l \Sigma_i)) \sim 2$.

Proof. Take a *bi*-infinite sequence

$$\cdots < t_{-k} < t_{-k+1} < \cdots < t_0 < t_1 < \cdots < t_k < \cdots$$

such that $|t_k - t_{k+1}| < \alpha$ with α is very small, then

$$\Lambda \subset \bigcup_{k=-\infty}^{+\infty} \phi^{[t_k, t_{k+1}]}(\Lambda \cap (\bigcup_{i=1}^l \Sigma_i)) := \bigcup_{k=-\infty}^{+\infty} A_k$$

since $HD(\Lambda) \sim 3$ and $HD(\Lambda) \leq \sup_k HD(A_k)$. Then there exists k_0 such that

$$HD(A_{k_0}) \sim 3.$$

For α very small, the map

$$\begin{aligned} \psi : \left(\Lambda \cap \left(\bigcup_{i=1}^l \Sigma_i \right) \right) \times [t_k, t_{k+1}] &\longrightarrow A_k \text{ defined by} \\ (x, t) &\longmapsto \phi^t(x) \end{aligned}$$

is Lipschitz. We see this since $\psi = \phi / (\Lambda \cap (\bigcup_{i=1}^l \Sigma_i)) \times [t_k, t_{k+1}]$ where $\phi / (\bigcup_{i=1}^l \Sigma_i) \times [t_k, t_{k+1}]$ is a diffeomorphism, for $|t_{k+1} - t_k| < \alpha$ and α very small. Therefore, $HD(\Lambda) \sim HD(A_{k_0}) \leq HD\left(\left(\Lambda \cap \bigcup_{i=1}^l \Sigma_i\right) \times [t_{k_0}, t_{k_0+1}]\right)$. Call $I_{k_0} = [t_{k_0}, t_{k_0+1}]$, we have the following inequality :

$$HD\left(\Lambda \cap \bigcup_{i=1}^l \Sigma_i \times I_{k_0}\right) \leq HD\left(\Lambda \cap \bigcup_{i=1}^l \Sigma_i\right) + D(I_{k_0}),$$

where D is a upper box counting dimension of I_{k_0} , is easy to see that $D(I_{k_0}) = 1$ (cf. [Fal85]). Thus,

$$HD(\Lambda) \sim HD\left(\left(\Lambda \cap \bigcup_{i=1}^l \Sigma_i\right) \times I_{k_0}\right) \leq HD\left(\Lambda \cap \bigcup_{i=1}^l \Sigma_i\right) + D(I_{k_0}) = HD\left(\Lambda \cap \bigcup_{i=1}^l \Sigma_i\right) + 1.$$

Hence, $HD(\Lambda \cap \bigcup_{i=1}^l \Sigma_i) \sim HD(\Lambda) - 1 \sim 2$. \square

4 Markov and Lagrange Spectrum For Geodesic Flow

Let M be a complete noncompact surface M such that the Gaussian curvature is bounded between two negative constants and the Gaussian volume is finite. Let ϕ be the vector field defining the geodesic flow. In this section we prove that the dynamical Lagrange and Markov spectrum has interior non empty, for vector field and geodesic flow close to ϕ .

4.1 The Interior of Spectrum for Perturbations of ϕ

The objective of this section is to prove the following theorems.

Theorem 1. *Let M be as above, let ϕ be the geodesic flow, then there is X a vector field sufficiently close to ϕ such that*

$$\text{int}M(f, X) \neq \emptyset \text{ and } \text{int}L(f, X) \neq \emptyset$$

for a dense and C^2 -open subset \mathcal{U} of $C^2(SM, \mathbb{R})$. Moreover, the above holds for a neighborhood of $\{X\} \times \mathcal{U}$ in $\mathfrak{X}^1(SM) \times C^2(SM, \mathbb{R})$, where $\mathfrak{X}^1(SM)$ is the space of C^1 vector field on SM .

Theorem 2. *Let M be as above, let ϕ be the geodesic flow, then there is X a vector field sufficiently close to ϕ such that*

$$\text{int}M(f \circ \pi, X) \neq \emptyset \text{ and } \text{int}L(f \circ \pi, X) \neq \emptyset$$

for a dense and C^2 -open subset \mathcal{V} of $C^2(M, \mathbb{R})$. Moreover, the above holds for a neighborhood of $\{X\} \times \mathcal{V}$ in $\mathfrak{X}^1(SM) \times C^2(M, \mathbb{R})$, where $\mathfrak{X}^1(SM)$ is the space of C^1 vector field on SM .

To prove this theorems we use the results of section 3 and [MRn13] and a construction for obtain the property V (cf. section 5.5 and [MY01]).

In section 3 it was proven that there are a finite number of C^1 -GCS, Σ_i pairwise disjoint and such that the Poincaré map \mathcal{R} (map of first return) of $\Xi := \bigcup_{i=1}^l \Sigma_i$

$$\mathcal{R}: \Xi \rightarrow \Xi$$

satisfies:

- $\bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n}(\Xi) := \Delta$ is hyperbolic set for \mathcal{R} .
- $HD(\Delta) \sim 2$.

We can assume without loss of generality that the GCS Σ_i are C^∞ -GCS.

4.1.1 The Family of Perturbation

Now we describe the family of perturbations of \mathcal{R} for which we can find the property V (cf. subsection 5.5 and [MY01]). Let R be a Markov partition of $\Delta = \bigcap_{n \in \mathbb{Z}} \mathcal{R}^{-n}(\Xi)$. Is selected once and for all a constant $c_0 > 1$. For all $0 < \rho < 1$, then we denote $R(\rho)$ the set of words \underline{a} of R such that $c_0^{-1}\rho \leq |I(\underline{a})| \leq c_0\rho$. We consider a partition \tilde{R}_1 of Δ in rectangle whose two sides are approximately sized $\rho^{2/m}$. A rectangle denotes here a part of Δ consisting of points with itinerary prescribed for a certain time interval; a word of R , which prescribes the route is associated with said rectangle. Among the rectangles on \tilde{R}_1 preserves only those for which no word in $R(\rho^{\frac{1}{2m}})$ appears no more than once in the associated word. Called R_1 the set of associated words.

For each $\underline{a} \in R_1$ denotes $R(\underline{a})$ the associated rectangle; construct a vector field $X_{\underline{a}}$ having the following properties:

- On $R(\underline{a})$, $X_{\underline{a}}$ is constant, the size of the order of approximately $\rho^{1+1/m}$ directed the unstable direction;
- $X_{\underline{a}}$ is the size $\rho^{1/m}$ in the $C^{m/2}$ -topology.

Clearly, the condition latter ensures that time one of the flow of X is, if m is large and ρ small, in a neighborhood of id_Ξ beforehand prescribed in C^∞ -topology.

We equip $\Omega = [-1, +1]^{R_1}$ the normalized Lebesgue measure; for $\underline{w} \in \Omega$. Let

$$X_{\underline{w}} = -c_X \sum_{\underline{a}} w(\underline{a}) X_{\underline{a}},$$

$$\mathcal{R}^{\underline{w}} = \mathcal{R} \circ \Phi^{\underline{w}},$$

where $\Phi^{\underline{w}}$ denotes the time one of the flow $X_{\underline{w}}$. Note that the “sum” in the definition of $X_{\underline{w}}$ has at each point of $\bigcup R(\underline{a})$ at most one nonzero term.

4.1.2 Realization of the Perturbation

In [MY10] was proven that for c_X large enough, appropriately chosen, there are many parameters \underline{w} such that $(\mathcal{R}^{\underline{w}}, \Delta_{\underline{w}})$ has the property V (cf. subsection 5.5), where $\Delta_{\underline{w}}$ is the continuation of the hyperbolic set Δ for $\mathcal{R}^{\underline{w}}$.

Lemma 15. *Let \mathcal{R} be as above, $\underline{w} \in \Omega$ and the vector field $X_{\underline{w}}$, then there is a vector field $G_{\underline{w}}$ sufficiently close to ϕ such that $\mathcal{R}^{\underline{w}}$ is the Poincaré map, (map of first return) to Ξ by the flow of $G_{\underline{w}}$.*

Proof. The argument is made on the $R(\underline{a})$ with $R(\underline{a}) \in R_1$. Fix $\underline{w} \in \Omega = [-1, +1]^{R_1}$, $\underline{w} = (w(\underline{a}))_{\underline{a} \in R_1}$. Then, on sufficiently small neighborhood of $R(\underline{a})$ in Ξ is defined the vector field $-C_X w(\underline{a}) X_{\underline{a}} := Y_{\underline{a}}$. Now we can extend this vector field in a neighborhood of $R(\underline{a})$ in SM as follows: Suppose that $R(\underline{a}) \subset \Sigma \in \Xi$. Let $\beta_{\underline{a}} > 0$ such that $\phi^t(V_{\underline{a}}) \cap (\Xi \setminus \Sigma_i) = \emptyset$ for all $t \in [0, \beta_{\underline{a}})$, where $V_{\underline{a}} \supset R(\underline{a})$ neighborhood of $R(\underline{a})$ in Σ and such that $Y_{\underline{a}}$ is defined and $Y_{\underline{a}} = 0$ in $\Sigma \setminus V_{\underline{a}}$. Put $\tilde{V}_{\underline{a}} := \phi^{[0, \beta_{\underline{a}})}(V_{\underline{a}})$ a neighborhood of $R(\underline{a})$ in SM . This neighborhood can be seen as $V_{\underline{a}} \times [0, \beta_{\underline{a}})$. Define the vector field $\tilde{Y}_{\underline{a}}$ on $\tilde{V}_{\underline{a}}$ by $\tilde{Y}_{\underline{a}}(\phi^t(z)) = D\phi_z^t(Y_{\underline{a}}(z))$. Let $\varphi_{\underline{a}}$ be a smooth real function defined in $V_{\underline{a}} \times [0, \beta_{\underline{a}})$ such that

$$\varphi_{\underline{a}} = \begin{cases} 1 & \text{in } V_{\underline{a}} \times [0, \frac{\beta_{\underline{a}}}{4}); \\ 0 & \text{in } V_{\underline{a}} \times [\frac{\beta_{\underline{a}}}{2}, \beta_{\underline{a}}) \end{cases}.$$

Put the vector field $Z_{\underline{a}} = \varphi_{\underline{a}} \tilde{Y}_{\underline{a}}$ defined in $V_{\underline{a}} \times [0, \beta_{\underline{a}})$. Note that by definition

$$Z_{\underline{a}} = 0 \quad \text{in } \Sigma \times [0, \beta_{\underline{a}}) \setminus V_{\underline{a}} \times [0, \beta_{\underline{a}}/2). \quad (14)$$

We will describe the relation between the diffeomorphism of time one of $Z_{\underline{a}}$ say $\Phi_{Z_{\underline{a}}}$ and the diffeomorphism of time one of $Y_{\underline{a}}$, say $\Phi_{Y_{\underline{a}}}$ in $V_{\underline{a}}$. In fact:

Let $0 \leq t_0 < \frac{\beta_{\underline{a}}}{4}$, $z \in V_{\underline{a}}$, take $\alpha(t)$ an integral curve of the vector field $Y_{\underline{a}}$ with $\alpha(0) = z$ contained in $V_{\underline{a}}$. Then we consider the curve $\eta(t) = \phi^{t_0}(\alpha(t))$. Therefore, as $\varphi_{\underline{a}}(\eta(t)) = 1$, we have

$$\begin{aligned} \eta'(t) &= D\phi_{\alpha(t)}^{t_0}(\alpha'(t)) = D\phi_{\alpha(t)}^{t_0}(Y_{\underline{a}}(\alpha(t))) \\ &= \tilde{Y}_{\underline{a}}(\phi^{t_0}(\alpha(t))) = \tilde{Y}_{\underline{a}}(\eta(t)) = \varphi_{\underline{a}} \tilde{Y}_{\underline{a}}(\eta(t)) = Z_{\underline{a}}(\eta(t)). \end{aligned}$$

So, we have the following equation

$$\Phi_{Z_{\underline{a}}}(\phi^{t_0}(z)) = \phi^{t_0}(\Phi_{Y_{\underline{a}}}(z)). \quad (15)$$

Varying $t_0 \in [0, \frac{\beta_a}{4})$ and differentiating the equation (15) with respect to t_0 , implies

$$D(\Phi_{Z_a})_{\phi^t(z)}(\phi(\phi^t(z))) = \phi(\phi^t(\Phi_{Y_a}(z))) \quad \text{for } z \in V_a,$$

where ϕ is the vector field defining the geodesic flow. Note that $\Phi_{Z_a} = \Phi_{Y_a}$ on V_a , call $h_a := \Phi_{Z_a}$. Then

$$D(h_a)_{\phi^t(z)}(\phi(\phi^t(z))) = \phi(\phi^t(h_a(z))) \quad \text{for } (z, t) \in V_a \times [0, \beta_a/4). \quad (16)$$

Define the vector field $G_a(x) = (Dh_a)_{h_a(x)}^{-1}(\phi(h_a(x)))$ for $x \in \tilde{V}_a$, then by (15) and (16) we have

$$G_a(x) = \phi(x) \quad \text{for any } x \in V_a \times [0, \beta_a/4).$$

And by (14)

$$G_a(x) = \phi(x) \quad \text{for any } x \in \Sigma \times [0, \beta_a) \setminus V_a \times [0, \beta_a/2).$$

These last two relations implies that G_a is a smooth vector field that coincides with ϕ outside of $V_a \times [\frac{\beta_a}{4}, \frac{\beta_a}{2})$.

Let $\beta(t)$ be the geodesic $\phi^t(h_a(z))$ with $z \in V_a$, define $\alpha(t) = h_a^{-1}(\beta(t))$, then

$$\begin{aligned} \alpha'(t) &= (Dh_a)_{\beta(t)}^{-1}(\beta'(t)) = (Dh_a)_{\beta(t)}^{-1}(\phi(\beta(t))) \\ &= (Dh_a)_{h_a(\alpha(t))}^{-1}(\phi(h_a(\alpha(t)))) = G_a(\alpha(t)) \end{aligned}$$

this $\alpha(t)$ is an integral curve of G_a in z .

Since $G_a = \phi$ outside of neighborhood of $V_a \times [\frac{\beta_a}{4}, \frac{\beta_a}{2})$ in SM , then the integral curve of vector field G_a passing by z coincides with the orbit of geodesic flow of $h_a(z)$ outside of neighbourhood $V_a \times [\frac{\beta_a}{4}, \frac{\beta_a}{2})$ of z in SM .

In particular, denoting $\mathcal{R}_{G_a} : \Xi \rightarrow \Xi$ the Poincaré map of vector field G_a , we have that if $z \in R(a)$ and $h_a(z) \in (R(a))$ then

$$(\mathcal{R} \circ h_a)(z) = \mathcal{R}_{G_a}(z).$$

Now for each $R(a) \in R_1$ we can assume that $\text{supp} Y_a := \overline{\{x : Y_a(x) \neq 0\}}$ the support vector field Y_a , are pairwise disjoint. So for each $R(a) \in R_1$ the vector field G_a can be constructed such that the sets $\text{supp}_\phi G_a = \overline{\{z \in SM : G_a(z) \neq \phi(z)\}}$ are disjoint. Define the smooth vector field

$$G_w(z) = \begin{cases} G_a(z) & \text{if } z \in \text{supp}_\phi G_a; \\ \phi(z) & \text{otherwise} \end{cases}$$

than satisfies $\mathcal{R}^w = \mathcal{R} \circ \Phi^w = \mathcal{R}_{G_w}$, where \mathcal{R}_{G_w} is the Poincaré map of vector field G_w . \square

Remark 9. Note that since X_a is small size, then taking β_a sufficiently small. Then G_a can be constructed close to ϕ . Therefore G_w is close to ϕ .

4.1.3 Combinatorial Arguments

The following Lemma is combinatorial and will be used to show the Lemma 17.

Lemma 16. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix such that $a_{ij} \in \{0, 1\}$ for any i, j and $|\{(i, j) : a_{ij} = 1\}| \geq \frac{99}{100}n^2$, then $\text{tr}(A^k) \geq \left(\frac{n}{2}\right)^k$ for all $k \geq 2$. Moreover, there is a set $Z \subset \{1, 2, \dots, n\}$ with $|Z| \geq \frac{4n}{5}$ such that, for any $k \geq 2$ and any $i, j \in Z$, we have*

$$(A^k)_{ij} \geq \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1}.$$

Remember that if $B = (b_{ij})_{1 \leq i, j \leq n}$ is a square matrix, then $\text{tr}(B) = \sum_{i=1}^n b_{ii}$ denotes the trace of B .

Proof. There is $X \subset \{1, 2, \dots, n\}$ with $|X| \geq \frac{9n}{10}$ such that, for any $i \in X$, $|\{j \leq n : a_{ij} = 1\}| \geq \frac{9n}{10}$. Indeed, if there are more than $\frac{n}{10}$ lines in the matrix, each with at least $\frac{n}{10}$ null entries, then the number of null entries of the matrix is greater than $\frac{n^2}{100}$, and so $|\{(i, j) : a_{ij} = 1\}| < n^2 - \frac{n^2}{100} = \frac{99n^2}{100}$ which is a contradiction. Analogously, there is $Y \subset \{1, 2, \dots, n\}$ with $|Y| \geq \frac{9n}{10}$ such that, for any $j \in Y$, $|\{i \leq n : a_{ij} = 1\}| \geq \frac{9n}{10}$. Let $Z = X \cap Y$; we have $|Z| \geq \frac{9n}{10} + \frac{9n}{10} - n = \frac{4n}{5}$. If $i, j \in Z$, then

$$(A^2)_{ij} = \sum_{r=1}^n a_{ir}a_{rj} = \sum_{r \in A_i \cap B_j} a_{ir}a_{rj} = |A_i \cap B_j| \geq \frac{9n}{10} + \frac{9n}{10} - n = \frac{4n}{5},$$

where $A_i = \{j \leq n : a_{ij} = 1\}$ and $B_j = \{i \leq n : a_{ij} = 1\}$. We will show by induction that if $i, j \in Z$, then

$$(A^k)_{ij} \geq \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} \text{ for all } k \geq 2.$$

In fact, the case $k = 2$ was proved above and, given $k \geq 2$ for which the statement is true, we have

$$\begin{aligned} (A^{k+1})_{ij} &= \sum_{r=1}^n (A^k)_{ir} \cdot a_{rj} \geq \sum_{r \in Z} (A^k)_{ir} \cdot a_{rj} \geq |Z \setminus \{r \in Z : a_{rj} = 0\}| \cdot \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} \\ &\geq \left(\frac{4n}{5} - \frac{n}{10}\right) \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} > \frac{4}{5} \left(\frac{3}{5}\right)^{k-1} \cdot n^k, \end{aligned}$$

since $|\{r \in Z : a_{rj} = 0\}| \leq \frac{n}{10}$.

Thus, for all $k \geq 2$

$$\text{tr}(A^k) \geq \sum_{i \in Z} (A^k)_{ii} \geq \frac{4n}{5} \cdot \frac{4}{5} \left(\frac{3}{5}\right)^{k-2} \cdot n^{k-1} > \left(\frac{3}{5}\right)^k \cdot n^k > \left(\frac{n}{2}\right)^k.$$

□

Remark 10. Suppose that the matrix A as in Lemma 16, is the matrix of transitions for a regular Cantor set K with Markov partition $R = \{R_1, R_2, \dots, R_n\}$ defined by an expansive map ψ satisfying $C^{-1}/\varepsilon < |\psi'(x)| < C/\varepsilon, \forall x \in \cup_{i \leq n} R_i$, for a suitable constant C (with $\log C \ll \log \varepsilon^{-1}$). From Lemma 16 we get a set Z of indices with $|Z| \geq \frac{4n}{5}$. Fix indices $\tilde{i}, \tilde{j} \in Z$ such that $a_{\tilde{i}\tilde{j}} = 1$. Consider a Markov partition for ψ^{k+2} corresponding to the words in the set

$$X = \{\tilde{j}r_1r_2 \cdots r_k\tilde{i} : r_i \leq n \text{ and } a_{\tilde{j}r_1} = a_{r_1r_2} = \cdots = a_{r_{k-1}r_k} = a_{r_k\tilde{i}} = 1\}.$$

By Lemma 16, $|X| = (A^{k+1})_{\tilde{i}\tilde{j}} \geq \frac{4}{5} \left(\frac{3}{5}\right)^{k-1} \cdot n^k > \left(\frac{n}{2}\right)^k$, since $a_{\tilde{i}\tilde{j}} = 1$ any transition between two words in X is admissible.

Consider the regular Cantor set

$$\tilde{K} := \{\alpha_1\alpha_2\alpha_3 \dots \mid \alpha_i \in X, \forall i \geq 1\} \subset K.$$

Take k large. We have $|(\psi^{k+2})'| < \left(\frac{C}{\varepsilon}\right)^{k+2}$, and this implies that

$$\begin{aligned} HD(\tilde{K}) &> \frac{\log \left(\frac{N}{2}\right)^k}{\log \left(\frac{C}{\varepsilon}\right)^{k+2}} = \frac{k}{k+2} \cdot \frac{\log n - \log 2}{\log C - \log \varepsilon} = (1-o(1)) \frac{\log n}{\log(\varepsilon^{-1})} = (1-o(1)) \frac{\log n}{\log(C^{-1}/\varepsilon)} \geq \\ &\geq (1-o(1))HD(K). \end{aligned}$$

It follows that $HD(\tilde{K}) \sim HD(K) \sim \frac{\log n}{\log(\varepsilon^{-1})}$.

The following Lemma says as is the behaviour of the horseshoe Δ when it is intersected by a finite number of C^1 -curves.

Lemma 17. Intersection of curves with Δ

Let $\alpha = \{\alpha_i : [0, 1] \rightarrow \Xi, i \in \{1, \dots, m\}\}$ be a finite family of C^1 -curves. Then for all $\epsilon > 0$ there are sub-horseshoes $\Delta_\alpha^s, \Delta_\alpha^u$ of Δ such that $\Delta_\alpha^{s,u} \cap \alpha_i([0, 1]) = \emptyset$ for any $i \in \{1, \dots, m\}$ and

$$HD(K_\alpha^s) \geq HD(K^s) - \epsilon \text{ and } HD(K_\alpha^u) \geq HD(K^u) - \epsilon,$$

where K_α^s, K^s are regular Cantor sets that describe the geometry transverse of the unstable foliation $W^u(\Delta_\alpha^s), W^u(\Delta)$ respectively, and K_α^u, K^u are regular Cantor set that describe the geometry transverse of the stable foliation $W^s(\Delta_\alpha^u), W^s(\Delta)$, respectively (cf. subsection 5.4).

We will prove this result for the stable Cantor set. For the unstable Cantor set the proof is analogous.

Before starting the proof of the Lemma we introduce some definitions and remarks.

Let us fix a Markov partition R of Δ as above and a point $p \in \Delta$. Given $R(\underline{a}) \in R$ for $\underline{a} = (a_{i_1}, \dots, a_{i_r})$ denote $|(a_{i_1}, \dots, a_{i_r})|$ the diameter of the projection on W_{loc}^s of $R(\underline{a})$ by the foliation \mathcal{F}^u (cf. the construction of $K^s(p)$ in the subsection 5.4 at the appendix). Fix a_r, a_s such that the pair (a_r, a_s) is admissible. Let $\epsilon > 0$, we have the following definition.

Definition 2. A piece $(a_{i_1}, \dots, a_{i_k})$ (in the construction of K^s) is called an ϵ -piece if

$$|(a_{i_1}, \dots, a_{i_k})| < \epsilon \text{ and } |(a_{i_1}, \dots, a_{i_{k-1}})| \geq \epsilon.$$

Put

$$X_\epsilon = \{\epsilon\text{-piece } (a_{i_1}, \dots, a_{i_k}) : i_1 = s \text{ and } i_k = r\} = \{\theta_1, \dots, \theta_N\}.$$

Notice that $\theta_i \theta_j$ is a admissible word for every $i, j \leq N$. We define

$$K(X_\epsilon) := \{\theta_{j_1} \theta_{j_2} \dots \theta_{j_k} \dots | \theta_{j_i} \in X_\epsilon, \forall i \geq 1\} \subset K^s.$$

Notice that $N \sim \epsilon^{-d_s}$ where $d_s = HD(K^s)$, and so $HD(K(X_\epsilon))$ is close to $HD(K^s)$ provided ϵ is small enough.

Dividing the curves in smaller curves if necessary, we can assume that the finite family α is formed by curves that are graphs of C^1 -functions of $W^s(\Delta)$ on $W^u(\Delta)$ or from $W^u(\Delta)$ on $W^s(\Delta)$.

Denote by I_{θ_i} the interval associated with θ_i in the construction of K^s . There is a constant $C > 1$ (which depends on the geometry of the horseshoe Δ , but not on ϵ) such that

$$C^{-1}\epsilon < |I_{\theta_i}| < C\epsilon.$$

For each I_{θ_i} , with $\theta_i = (a_{i_1}, \dots, a_{i_k})$, we associate the interval $I'_{\theta_i^t}$ corresponding to the transposed sequence $\theta_i^t = (a_{i_k}, \dots, a_{i_1})$ in the construction of K^u (unstable Cantor set) - by an abuse of language, we will say that the interval $I'_{\theta_i^t}$ is the transposed interval of I_{θ_i} (and vice-versa). Then, since Δ is horseshoe there exists $\beta \geq 1$ (which depends on the geometry of the horseshoe Δ , but not on ϵ or k) such that

$$C^{-1}|I_{\theta_i}|^\beta < |I'_{\theta_i^t}| < C|I_{\theta_i}|^{1/\beta}.$$

Remark 11. In the conservative case (i.e. when the horseshoe is defined by a diffeomorphism which preserves a smooth measure), the above inequality holds with $\beta = 1$. This will be useful in the section 4.2.2.

Proof of Lemma 17.

- First case. (Graph of a C^1 -function from $W^s(\Delta)$ on $W^u(\Delta)$). In this case, consider the image P of I_{θ_i} by this function. Then C and ϵ can be taken such that $|P| \leq C^2\epsilon$. Let P' , the smallest interval of the construction of K^u containing P . Then, if $J \in W^s(\Delta)$ is the transposed interval of P' , we have $|J| \leq (C^2\epsilon)^{1/\beta}$. Then

$$\#\{I_{\theta_j} : I_{\theta_j} \subset J\} \leq C \left(\frac{(C^2\epsilon)^{1/\beta}}{\epsilon} \right)^{d_s} = \tilde{C} \epsilon^{d_s(1/\beta-1)}.$$

Thus,

$$\#\{(I_{\theta_i}, I'_{\theta_i^t}) : I_{\theta_i} \times I'_{\theta_i^t} \text{ intersects the curve}\} \leq \epsilon^{-d_s} \tilde{C} \epsilon^{d_s(1/\beta-1)} = \tilde{C} \epsilon^{d_s(1/\beta-2)} \ll \epsilon^{-2d_s}.$$

- Second case. (Graph of a C^1 -function from $W^u(\Delta)$ on $W^s(\Delta)$). In this case, consider the image J' of $I'_{\theta_i^t}$. Then, $|J'| \leq c|I'_{\theta_i^t}| \leq c(C\epsilon)^{1/\beta}$, (J' is the image of $I'_{\theta_i^t}$ by a C^1 -function), so we have analogously

$$\#\{I_{\theta_i} : I_{\theta_i} \subset J'\} \leq \hat{C}\epsilon^{d_s(1/\beta-1)}.$$

And

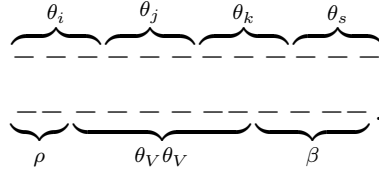
$$\#\{(I_{\theta_i}, I'_{\theta_j^t}) : I_{\theta_i} \times I'_{\theta_j^t} \text{ intersects the curve}\} \leq \epsilon^{-d_s} \tilde{C} \epsilon^{d_s(1/\beta-1)} = \hat{C} \epsilon^{d_s(1/\beta-2)} \ll \epsilon^{-2d_s}.$$

Note that $\epsilon^{-2d_s} \sim N^2 = \text{total number of transitions } \theta_i \theta_j$.

We say that $\theta_U \theta_V$ is a prohibited transition iff some curve of the family α intersects the rectangle $I_{\theta_U} \times I'_{\theta_V}$.

Consider the admissible word $\theta_i \theta_j \theta_k \theta_s$ with $\theta_i, \theta_j, \theta_k, \theta_s \in X_\epsilon$. This word generates an interval of size of the order of ϵ^4 in the construction of K^s .

We say that $\theta_i \theta_j \theta_k \theta_s$ is a prohibited word, if within there is a prohibited transition $\theta_U \theta_V$



Denote by PW the set of the prohibited words $\theta_i \theta_j \theta_k \theta_s$. We want to now estimate $|PW|$. In fact: $|I_\rho| |I_\beta| \sim \epsilon^2 \sim 2^{-2n}$, then there is $t \leq 2n$ such that $|I_\rho| \sim 2^{-t}$ and $|I_\beta| \sim 2^{t-2n}$. Thus, $\#\{I_\rho\} \sim (2^{-t})^{-d_s} = 2^{td_s}$ and $\#\{I_\beta\} \sim (2^{-(2n-t)})^{-d_s} = 2^{(2n-t)d_s}$. Therefore for some constant $\tilde{C} > 1$ (as in the first part of the proof), we have that

$$|PW| \leq \tilde{C} \cdot (2n) \cdot 2^{td_s} 2^{(2n-t)d_s} \epsilon^{d_s(1/\beta-2)} \leq 2\tilde{C} \log \epsilon^{-1} \epsilon^{d_s(1/\beta-4)} \ll \epsilon^{-4d_s}$$

the last inequality follows from $2\tilde{C}(\log \epsilon^{-1}) \epsilon^{d_s/\beta} \ll 1$.

Then, the total of prohibited words $\theta_i \theta_j \theta_k \theta_s$ is much less than $\epsilon^{-4d_s} \sim N^4$, the total number of words $\theta_i \theta_j \theta_k \theta_s$.

Consider $A = (a_{(i,j)(k,s)})$ for $(i,j), (k,s) \in \{1, \dots, N\}^2$ the matrix defined by

$$a_{(i,j)(k,s)} = \begin{cases} 1 & \text{if } \theta_i \theta_j \theta_k \theta_s \text{ is not prohibited;} \\ 0 & \text{if } \theta_i \theta_j \theta_k \theta_s \text{ is prohibited for some } \theta_U \theta_V. \end{cases}$$

Put $\tilde{\theta}_{ij} = \theta_i \theta_j$ for $i, j \leq N$. Define \tilde{K} the regular Cantor set

$$\tilde{K} := \{\tilde{\theta}_{i_1 j_1} \tilde{\theta}_{i_2 j_2} \cdots \tilde{\theta}_{i_n j_n} \cdots | a_{(i_k, j_k)(i_{k+1}, j_{k+1})} = 1, \forall k \geq 1\} \subset K^s.$$

By the previous discussion we have $\#\{a_{(i,j)(k,s)} : a_{(i,j)(k,s)} = 1\} \geq \frac{99}{100}(N^2)^2$, so by the Remark 10 we have $HD(\tilde{K}) \sim HD(K(X_\epsilon)) \sim HD(K^s)$. Consider the sub-horseshoe of Δ defined by

$$\Delta_\alpha^s := \bigcap_{n \in \mathbb{Z}} \mathcal{R}^n \left(\bigcup_{(i,j),(k,s) \in \{1,2,\dots,N\}^2, a_{(i,j)(k,s)}=1} (R(\tilde{\theta}_{ij}) \cap \mathcal{R}^{-1}(R(\tilde{\theta}_{ks}))) \right),$$

where $R(\tilde{\theta}_{ij})$ is the rectangle associated to the word $\tilde{\theta}_{ij}$.

Then the stable regular Cantor set K_α^s describing the transverse geometry of the unstable foliation $W^u(\Delta_\alpha^s)$ is equal to \tilde{K} . Then by the above discussion we have that

$$HD(K_\alpha^s) \sim HD(K^s),$$

and by definition of Δ_α^s we have that $\Delta_\alpha^s \cap \alpha_i = \emptyset, \forall i \leq m$. This concludes the proof. \square

4.1.4 Regaining the Spectrum

Given $F \in C^0(SM, \mathbb{R})$, we can define the function $\max F_\phi: \Xi \rightarrow \mathbb{R}$ by

$$\max F_\phi(x) := \max_{t_-(x) \leq t \leq t_+(x)} F(\phi^t(x))$$

where $t_-(x), t_+(x)$ are such that $\mathcal{R}^{-1}(x) = \phi^{t_-(x)}(x)$ and $\mathcal{R}(x) = \phi^{t_+(x)}(x)$.

Note that this definition depends on the geodesic flow ϕ^t , or equivalently the vector field ϕ . Note also that $\max F_\phi$ is always a continuous function, but even if F is C^∞ , $\max F_\phi$ can be only a continuous function. In what follows we try to give some “differentiability” to $\max F_\phi$ at least for $F \in C^2(SM, \mathbb{R})$ (see Lemma 18).

Consider the set

$$\mathcal{O} = \{F \in C^\infty(SM, \mathbb{R}) : \max F_\phi(x) = F(\phi^{t(x)}(x)) \text{ and } t_-(x) < t(x) < t_+(x) \text{ for all } x \in D_\mathcal{R}\},$$

where $D_\mathcal{R}$ is the domain of \mathcal{R} .

Is easy to see that the set \mathcal{O} is open and dense subset of $C^\infty(SM, \mathbb{R})$.

Remark 12. Let $x \in \text{int}(\Sigma)$ with $\Sigma \in \Xi$ such that $\mathcal{R}(x) = \phi^{t_+(x)}(x) \in \text{int}(\Xi)$, by The Large Tubular Flow Theorem, there exists a neighborhood $U_x \subset \Sigma$ of x , a diffeomorphism $\varphi: U_x \times (-\epsilon, t_+(x) + \epsilon) \rightarrow \varphi(U_x \times (-\epsilon, t_+(x) + \epsilon)) \subset SM$ such that $D\varphi_{(z,t)}(0, 0, 1) = \phi(\varphi(z, t))$ for $(z, t) \in U_x \times (-\epsilon, t_+(x) + \epsilon)$. Moreover, as the elements of the Markov partition are disjoint, has small diameter and Δ is compact, then, we can suppose that there is a finite number an open set U_{x_i} such that $U_{x_i} \cap U_{x_j} = \emptyset$ and $\Delta \subset \bigcup U_{x_i}$ for some $x_i \in \Delta$. Denote $\varphi_i: U_{x_i} \times (-\epsilon, t_+(x_i) + \epsilon) \rightarrow \varphi_i(U_{x_i} \times (-\epsilon, t_+(x_i) + \epsilon)) \subset SM$ such that $(D\varphi_i)_{(z,t)}(0, 0, 1) = \phi(\varphi_i(z, t))$.

Remark 13. Let $F \in \mathcal{O}$, consider the function $f(x_1, x_2, x_3) = F \circ \varphi_i(x_1, x_2, x_3)$, we want to see the behaviour of the critical points of $F \circ \varphi_i|_{\{z\} \times (-\epsilon, t_+(z) + \epsilon)}$. Let δ be small regular value of $\frac{\partial f}{\partial x_3}(z_1, z_2, z_3)$, then $f_\delta(x_1, x_2, x_3) = f(x_1, x_2, x_3) - \delta x_3$ has 0 as regular value, so $\left(\frac{\partial f_\delta}{\partial x_3}\right)^{-1}(0) := S_\delta$ is a surface. We want that this surface does not contain an open consisting of orbits of the flow. Observe also that if $(0, 0, 1) \in T_z S_\delta$ for $z = (z_1, z_2, z_3)$,

then $D\left(\frac{\partial f_\delta}{\partial x_3}\right)_z(0,0,1) = 0$, this implies that $\frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0$ so, $z \in \left\{x : \frac{\partial^2 f_\delta}{\partial x_3^2}(x) = 0\right\}$. Thus, (z, t_0) is the critical point of $f_\delta|_{\{z\} \times (-\epsilon, t_z + \epsilon)}$, then $(z, t_0) \in S_\delta$. Moreover, if (z, t_0) is degenerate, then $(0,0,1) \in T_z S_\delta$.

Lemma 18. *There exists a dense $\mathcal{B}_\phi \subset C^\infty(SM, \mathbb{R})$ and C^2 -open such that given $\beta > 0$, then for any $F \in \mathcal{B}_\phi$ there are sub-horseshoe $\Delta_F^{s,u}$ of Δ with $HD(K_F^s) \geq HD(K^s) - \beta$, $HD(K_F^u) \geq HD(K^u) - \beta$ and a Markov partition $R_F^{s,u}$ of $\Delta_F^{s,u}$, respectively, such that the function $\max F_\phi|_{\Xi \cap R_F^{s,u}} \in C^1(\Xi \cap R_F^{s,u}, \mathbb{R})$, where $K_F^{s,u}$, $K^{s,u}$ as in Lemma 17.*

Proof. We prove the Lemma for Δ_F^s , for Δ_F^u is analogue. Let $F \in C^\infty(SM, \mathbb{R})$ and $f = F \circ \varphi_i$ as above, with $U_i \subset \Sigma$, we want to perturb f by a $f_\delta(x_1, x_2, x_3) = f(x_1, x_2, x_3) - \frac{\delta x_3^2}{2} - cx_3$ such that $\left\{z : \frac{\partial f_\delta}{\partial x_3}(z) = 0\right\}$ and $\left\{z : \frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0\right\}$ are surfaces which intersect transversely and put

$$J_\delta(x_i) := \left\{z : \frac{\partial f_\delta}{\partial x_3}(z) = 0\right\} \cap \left\{z : \frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0\right\}. \quad (17)$$

In fact:

Let δ be small regular value of $\frac{\partial^2 f}{\partial x_3^2}$, so $\left\{z : \frac{\partial^2 f_\delta}{\partial x_3^2}(z) = 0\right\} := \tilde{S}_\delta$ is surface for all $c \in \mathbb{R}$. Therefore, consider the function $\left(\frac{\partial f}{\partial x_3} - \delta x_3\right)|_{\tilde{S}_\delta}$, $\frac{\partial f}{\partial x_3} - \delta x_3$ restrict to \tilde{S}_δ . Thus taking c small a regular value of $\left(\frac{\partial f}{\partial x_3} - \delta x_3\right)|_{\tilde{S}_\delta}$, we have that f_δ satisfies (17). Therefore, by Remark 13 the surface $\left\{z : \frac{\partial f_\delta}{\partial x_3}(z) = 0\right\}$ does not contain an open consisting of orbits of the flow. Call α_{x_i} the projection of curves $J_\delta(x_i)$ along of the flow on Σ . Thus, considering the finite family of curve $\alpha := \{\alpha_{x_i}\}$, then by Lemma 17, given $\beta > 0$ small there is a sub-horseshoe Δ_α such that

$$HD(K_\alpha^s) \geq HD(K^s) - \beta \quad \text{and} \quad \Delta_\alpha \cap \alpha_i = \emptyset.$$

For $x \in \Delta_\alpha$ holds that the critical points of $f_\delta|_{\{x\} \times (-\epsilon, t_+(x) + \epsilon)}$ are non-degenerates and therefore finite. Thus, the critical points of f_δ are locally graphs of a finite number of functions ψ_j , that is, are locally $\{(x_1, x_2, \psi_j(x_1, x_2)); 1 \leq j \leq k\}$. Since we want the function $\max f_\delta$ be C^1 , we have to rid of the points (x_1, x_2) such that for $i \neq j$

$$f_\delta(x_1, x_2, \psi_i(x_1, x_2)) = f_\delta(x_1, x_2, \psi_j(x_1, x_2)).$$

In fact:

Let $g_{1j}(x_1, x_2) = f_\delta(x_1, x_2, \psi_1(x_1, x_2)) - f_\delta(x_1, x_2, \psi_j(x_1, x_2))$ for $j \neq 1$ and let $\gamma_1 > 0$ small regular value of g_{1j} for all $j \neq 1$. Take ξ_1 a C^∞ -function close to the constant function 0 and equal to $-\gamma_1$ in neighborhood of $\{z = \psi_1(x_1, x_2)\}$ and 0 outside. So, the function $f_\delta + \xi_1$ is close to f_δ . Now we define the function

$$g_{1j}^{\gamma_1}(x_1, x_2) = (f_\delta + \xi_1)(x_1, x_2, \psi_1(x_1, x_2)) - (f_\delta + \xi_1)(x_1, x_2, \psi_j(x_1, x_2)) = g_{1j}(x_1, x_2) - \gamma_1.$$

Put $f_1 := f_\delta + \xi_1$ and define $g_{2j}(x_1, x_2) = f_1(x_1, x_1, \psi_2(x_1, x_2)) - f_1(x_1, x_2, \psi_j(x_1, x_2))$ for $j \neq 2$ and let $\gamma_2 > 0$ small regular value of g_{2j} for all $j \neq 2$. Take ξ_2 a C^∞ -function close

to the constant function 0 and equal to $-\gamma_2$ in neighborhood of $\{z = \psi_2(x_1, x_2)\}$ and 0 outside. So, the function $f_1 + \xi_2$ is close to f_δ . Now we define the function

$$g_{2j}^{\gamma_2}(x_1, x_2) = (f_1 + \xi_2)(x_1, x_2, \psi_2(x_1, x_2)) - (f_1 + \xi_2)(x_1, x_2, \psi_j(x_1, x_2)) = g_{2j}(x_1, x_2) - \gamma_2.$$

Inductively, define $f_{s-1} = f_{s-2} + \xi_{s-1}$ and

$$g_{sj}(x_1, x_2) = f_{s-1}(x_1, x_2, \psi_s(x_1, x_2)) - f_{s-1}(x_1, x_2, \psi_j(x_1, x_2))$$

for $j \neq s$. Let $\gamma_s > 0$ small regular value of g_{sj} for all $j \neq s$. Take ξ_s a C^∞ -function close to the constant function 0 and equal to $-\gamma_s$ in neighborhood of $\{z = \psi_s(x_1, x_2)\}$ and 0 outside. So, the function $f_s := f_{s-1} + \xi_s$ is close to f_δ . Now we define the function

$$g_{sj}^{\gamma_s}(x_1, x_2) = f_s(x_1, x_2, \psi_s(x_1, x_2)) - f_s(x_1, x_2, \psi_j(x_1, x_2)) = g_{sj}(x_1, x_2) - \gamma_s.$$

Therefore, for each $s = 1, \dots, k-1$, we have that $\Gamma_s := \bigcup_{j \neq s} (g_{sj}^{\gamma_s})^{-1}(0)$ are a finite number

of curves in $U_i \subset \Sigma$. So, consider of finite family of curves $\Gamma = \bigcup_{s=1}^{k-1} \{\Gamma_s\}$, then by Lemma 17 there is a sub-horseshoe Δ_Γ of Δ_α such that

$$HD(K_\Gamma^s) \geq HD(K_\delta^s) - \beta \geq HD(K^s) - 2\beta \quad \text{and} \quad \Delta_\Gamma \cap \Gamma = \emptyset. \quad (18)$$

Consider the function $f_\delta + \xi^{k;i}$, where $\xi^{k;i} := \xi_1 + \dots + \xi_{k-1}$, then if $l < j$, we have

$$\begin{aligned} (f_\delta + \xi^{k;i})(x_1, x_2, \psi_j(x_1, x_2)) &= (f_\delta + \xi^{k;i})(x_1, x_2, \psi_l(x_1, x_2)) = \\ (f_\delta + \xi_1 + \dots + \xi_j)(x_1, x_2, \psi_j(x_1, x_2)) &= (f_\delta + \xi_1 + \dots + \xi_l)(x_1, x_2, \psi_l(x_1, x_2)) = \\ (f_\delta + \xi_1 + \dots + \xi_j)(x_1, x_2, \psi_j(x_1, x_2)) &= (f_\delta + \xi_1 + \dots + \xi_l + \dots + \xi_j)(x_1, x_2, \psi_l(x_1, x_2)) \\ &= g_{jl}^{\gamma_l}(x_1, x_2). \end{aligned}$$

Thus, if $(x_1, x_2) \in \Delta_\Gamma$, then

$$(f_\delta + \xi^{k;i})(x_1, x_2, \psi_j(x_1, x_2)) \neq (f_\delta + \xi^{k;i})(x_1, x_2, \psi_s(x_1, x_2)) \quad \text{for all } j \neq s. \quad (19)$$

As the tubes $U_i \times (-\epsilon, t_+(x_i) + \epsilon)$ are disjoint and each is defined a function $f_\delta + \xi^{k;i}$ close to $f = F \circ \varphi_i$. Then we have define a function $G \in \mathcal{O} \subset C^\infty(SM, \mathbb{R})$ close to F , with the following properties:

- $G = f_\delta + \xi^{k;i}$ on $U_i \times (-\epsilon, t_{x_i} + \epsilon)$ and $G = F$ outside of neighborhood of $\bigcup_i (U_i \times (-\epsilon, t_+(x_i) + \epsilon))$.

-Take a Markov partition R_Γ of Δ_Γ with diameter small, then $\max G_\phi|_{\Xi \cap R_\Gamma}$ is a C^1 -function.

The above is simply to observe the construction of G , and inequality (19), than implies the critical point of $G|_{\{x\} \times (-\epsilon, t_+(x) + \epsilon)}$ is a unique point for $x \in R_\Gamma$. Since $G \in \mathcal{O}$, we have the second item. Note also that by construction of G , we have that

$$\frac{\partial G}{\partial x_3}(x_1, x_2, \psi_k(x_1, x_2)) = 0 \quad \text{and} \quad \frac{\partial^2 G}{\partial x_3^2}(x_1, x_2, \psi_k(x_1, x_2)) \neq 0 \quad \text{in } U_i \cap R_\Gamma.$$

And this condition implies that, if H is C^2 close to G , then there exists $\tilde{\psi}_k$ C^1 -close to ψ_k and such that holds

$$\frac{\partial H}{\partial x_3}(x_1, x_2, \tilde{\psi}_k(x_1, x_2)) = 0 \text{ and } \frac{\partial^2 H}{\partial x_3^2}(x_1, x_2, \tilde{\psi}_k(x_1, x_2)) \neq 0 \text{ in } U_i \cap R_\Gamma.$$

This last condition implies that there is a single maximum of $H|_{\{(x_1, x_2)\} \times (-\epsilon, t_+(x_1, x_2) + \epsilon)}$. Thus $\max H_\phi|_{\Xi \cap R_\Gamma}$ is a C^1 -function. \square

Keeping the notation of the previous Lemma we have:

Corollary 2. *The above property is robust in the following sense: If X is a vector field C^1 -close to ϕ , then $\mathcal{B}_\phi = \mathcal{B}_X$ and for any $F \in \mathcal{B}_X$, holds that $\max F_X \in C^1(\Xi \cap R_F^{s,u}, \mathbb{R})$.*

Proof. As time $t_+(x)$ the time of return of $x \in \Xi$ to Ξ by the flow ϕ is bounded, then X is a vector field sufficiently C^1 -close to ϕ , we have that for $x \in \Xi$ the orbits $\phi^t(x)$ and $X^t(x)$ of the vector field ϕ and X by x respectively, are close.

Let $\varphi, \tilde{\varphi}$ be diffeomorphism given by The Large Tubular Flow Theorem (for ϕ and X , respectively, as above). Let $F \in C^\infty(SM, \mathbb{R})$, then $F \circ \varphi, F \circ \tilde{\varphi}$ are C^0 -close. Moreover, $\frac{\partial}{\partial x_3} F \circ \varphi(x) = DF_{\varphi(x)} D\varphi_x(0, 0, 1) = DF_{\varphi(x)}(\phi(\varphi(x)))$ and $\frac{\partial}{\partial x_3} F \circ \tilde{\varphi}(x) = DF_{\tilde{\varphi}(x)} D\tilde{\varphi}_x(0, 0, 1) = DF_{\tilde{\varphi}(x)}(X(\tilde{\varphi}(x)))$. Since ϕ is C^1 -close of X , then $F \circ \varphi|_{\{(x_1, x_2)\} \times (a, b)}$ is C^2 -close of $F \circ \tilde{\varphi}|_{\{(x_1, x_2)\} \times (a, b)}$.

Suppose that $F \in \mathcal{B}_\phi$, then by Lemma 18 there is a sub-horseshoe Δ_F of Δ and Markov partition R_F of Δ_F such that $\max F_\phi|_{\Xi \cap R_F}$ is C^1 . Thus by construction in proof of Lemma 18 we have that $\max F_X|_{\Xi \cap R_F}$ is C^1 . \square

The Lemma 18, has a version for functions in $C^2(M, \mathbb{R})$, in fact:

Lemma 19. *There exists a dense $\mathcal{C}_\phi \subset C^\infty(M, \mathbb{R})$ and C^2 -open such that given $\beta > 0$, then for any $f \in \mathcal{C}_\phi$ there are sub-horseshoe $\Delta_f^{s,u}$ of Δ with $HD(K_f^s) \geq HD(K^s) - \beta$, $HD(K_f^u) \geq HD(K^u) - \beta$ and a Markov partition $R_f^{s,u}$ of $\Delta_f^{s,u}$, respectively, such that the function $\max(f \circ \pi)_\phi|_{\Xi \cap R_f^{s,u}} \in C^1(\Xi \cap R_f^{s,u}, \mathbb{R})$, where $K_f^{s,u}, K^{s,u}$ as in Lemma 17.*

The proof of this lemma (at least the first part) is slightly different from the proof of Lemma 18, since the perturbations are made in M and not in SM .

Before to prove this lemma, we prove some auxiliary lemmas.

Remark 14. *As Δ is a hyperbolic set for \mathcal{R} , then the set of fixed points is finite. Thus, removing the Δ these fixed points, we still have a sub-horseshoe (we still call Δ) with almost the same Hausdorff dimension.*

Lemma 20. *Let U be an open set in SM such that $U \cap \bigcup_{(x,v) \in \Delta} \{\phi^t(x, v) : t \in (0, t_+(x, v))\} \neq \emptyset$, then there exists a dense $\mathcal{C}_U \subset C^\infty(M, \mathbb{R})$ and C^2 -open such that given $\beta > 0$, then for any $f \in \mathcal{C}_U$ there are sub-horseshoes $\Delta_f^{s,u}$ of Δ with $HD(K_f^s) \geq HD(K^s) - \beta$, $HD(K_f^u) \geq HD(K^u) - \beta$, such that if $(x, v) \in \Delta_f^{s,u}$ and $U \cap \{\phi^t(x, v) : t \in (0, t_+(x, v))\} \neq \emptyset$, then*

$$\#\{t : \phi^t(x, v) \in U \text{ and } t \text{ is a critical point of } f \circ \pi(\phi^t(x, v))\} < \infty. \quad (20)$$

Proof. Without loss of generality, we can assume that \overline{U} , of closure of U , is contained in the image of a parametrization. Thus, let $\varphi: V \subset \mathbb{R}^2 \rightarrow M$ a parametrization of M such that the set $\{\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\}$ is an orthonormal basis. Let I be an interval such that $\tilde{\varphi}: V \times I \rightarrow SM$ defined by $\tilde{\varphi}(x, y, z) = (\varphi(x, y), \cos z \frac{\partial\varphi}{\partial x} + \sin z \frac{\partial\varphi}{\partial y})$ is a parametrization of SM and $\overline{U} \subset \tilde{\varphi}(V \times I)$.

Let $f \in C^\infty(M, \mathbb{R})$, put $F = f \circ \pi$, then in local coordinates $F(x, y, z) = f \circ \pi \circ \tilde{\varphi}(x, y, z) = f(\varphi(x, y))$ and the vector field ϕ is $\phi(x, y, z) = (X_1(x, y, z), X_2(x, y, z), X_3(x, y, z))$.

Consider now the set

$$\begin{aligned} S &= \{(x, y, z) : \langle \nabla F(x, y, z), \phi(x, y, z) \rangle = 0\} \\ &= \left\{ (x, y, z) : \frac{\partial f}{\partial x} X_1(x, y, z) + \frac{\partial f}{\partial y} X_2(x, y, z) = 0 \right\} \quad \text{and} \\ H &= \{(x, y, z) : \langle Hess F(x, y, z) \phi(x, y, z), \phi(x, y, z) \rangle = 0\}, \end{aligned}$$

where $Hess F$ is the Hessian matrix of F , given by $Hess F(x, y, z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & 0 \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

So, $H = \{(x, y, z) : \frac{\partial^2 f}{\partial x^2} X_1^2 + 2 \frac{\partial^2 f}{\partial x \partial y} X_1 X_2 + \frac{\partial^2 f}{\partial y^2} X_2^2 = 0\}$. Now we would like perturb f so that the sets S and H to be manifolds that intersect transversely. In fact:

Since the vector field ϕ is transverse to fiber, then $X_1 \neq 0$ or $X_2 \neq 0$. Suppose that $X_1 \neq 0$, then put $f_\delta(x, y) = f(x, y) - \frac{\delta x^2}{2} - cx$. Taking δ small a regular value of

$$L(x, y, z) := \frac{\frac{\partial^2 f}{\partial x^2} X_1^2 + 2 \frac{\partial^2 f}{\partial x \partial y} X_1 X_2 + \frac{\partial^2 f}{\partial y^2} X_2^2}{X_1^2}.$$

Then, the set $H_\delta := \{L(x, y, z) = \delta\}$ is a regular surface for all c and

$$H_\delta = \left\{ (x, y, z) : \frac{\partial^2 f_\delta}{\partial x^2} X_1^2 + 2 \frac{\partial^2 f_\delta}{\partial x \partial y} X_1 X_2 + \frac{\partial^2 f_\delta}{\partial y^2} X_2^2 = 0 \right\}.$$

Consider now the function $G(x, y, z) := \frac{(\frac{\partial f}{\partial x} - \delta x) X_1 + \frac{\partial f}{\partial y} X_2}{X_1}$, let c be small a regular value of $G(x, y, z)|_{H_\delta}$, $G(x, y, z)$ restrict to H_δ . Thus, the regular surface

$$S_\delta := \{G(x, y, z) = c\} = \left\{ (x, y, z) : \frac{\partial f_\delta}{\partial x} X_1 + \frac{\partial f_\delta}{\partial y} X_2 = 0 \right\}.$$

Therefore, by the choice of c we have that $J_\delta := S_\delta \cap H_\delta$ is a finite number of curves. Call α_δ , the projections of the curves J_δ along of the flow on Ξ . Then, by Lemma 17, given $\beta > 0$ small there are a sub-horseshoes $\Delta_\delta^{s,u}$ such that $\Delta_\delta^{s,u} \cap \alpha_\delta = \emptyset$ and

$$HD(K_\delta^{s,u}) \geq HD(K^{s,u}) - \beta.$$

Thus, if $(x, v) \in \Delta_\delta^{s,u}$, then the critical points of $(f_\delta \circ \pi)(\phi^t(x, v))|_{\{t: \phi^t(x, v) \in U\}}$ are non-degenerates and therefore finite. Thus f_δ satisfies (20). Also, the above condition is open in $C^2(M, \mathbb{R})$. □

Corollary 3. *There exists a dense $\mathcal{C} \subset C^\infty(M, \mathbb{R})$ and C^2 -open, such that given $\beta > 0$, then for any $f \in \mathcal{C}$ there are sub-horseshoes $\Delta_f^{s,u}$ of Δ with $HD(K_f^s) \geq HD(K^s) - \beta$, $HD(K_f^u) \geq HD(K^u) - \beta$, such that if $(x, v) \in \Delta_f^{s,u}$, then*

$$\#\{t \in (0, t_+(x, v)) : t \text{ is a critical point of } f \circ \pi(\phi^t(x, v))\} < \infty.$$

Proof. Since $\bigcup_{x \in \Delta} \bigcup_{t \in [0, t_+(x)]} \phi^t(x)$ is a compact set, then there are a finite number of open set U_1, \dots, U_n in SM , such that

$$\bigcup_{x \in \Delta} \bigcup_{t \in [0, t_+(x)]} \phi^t(x) \subset \bigcup_{i=1}^n U_i.$$

Now, by Lemma 20 for each $i = 1, \dots, n$, there exists a dense $\mathcal{C}_{U_i} \subset C^\infty(M, \mathbb{R})$ and C^2 -open and satisfies all conditions of Lemma 20.

Put $\mathcal{C} = \bigcap_{i=1}^n \mathcal{C}_{U_i}$. Thus, give $\beta > 0$ and $f \in \mathcal{C}$, in particular $f \in \mathcal{C}_{U_1}$. Then, there are sub-horseshoes ${}_1\Delta_f^{s,u}$ of Δ with $HD({}_1K_f^s) \geq HD(K^s) - \beta/n$, $HD({}_1K_f^u) \geq HD(K^u) - \beta/n$ and we have (20) for U_1 .

Again, as $f \in \mathcal{C}_{U_2}$, the Lemma 20 implies that there are sub-horseshoes ${}_2\Delta_f^{s,u}$ of ${}_1\Delta_f^{s,u}$ with $HD({}_2K_f^s) \geq HD({}_1K_f^s) - \beta/n$, $HD({}_2K_f^u) \geq HD({}_1K_f^u) - \beta/n$ and we have (20) for U_1 and U_2 . In particular,

$$HD({}_2K_f^s) \geq HD(K^s) - 2\beta/n \quad \text{and} \quad HD({}_2K_f^u) \geq HD(K^u) - 2\beta/n.$$

Repeating the same argument several times, applying Lemma 20, we found sub-horseshoes ${}_n\Delta_f^{s,u}$ of Δ with $HD({}_nK_f^s) \geq HD(K^s) - \beta$, $HD({}_nK_f^u) \geq HD(K^u) - \beta$ and we have (20) for U_1, \dots, U_n . Thus we conclude the proof of Corollary. \square

Consider now the set \mathcal{M} of C^2 Morse's functions of M .

Let $f \in \mathcal{M} \cap \mathcal{C}$, since $\pi(\bigcup_{x \in \Delta} \bigcup_{x \in [0, t_+(x)]} \phi^t(x))$ is a compact set, then the set of critical points of f in $\pi(\bigcup_{x \in \Xi} \bigcup_{x \in [0, t(x)]} \phi^t(x))$ is finite and denote it by x_1^f, \dots, x_k^f . Call α_f the set of projections of the fibers $\pi^{-1}(x_i^f)$ along of the flow. Thus, applying the Lemma 17 a $\Delta_f^{s,u}$ (of the Corollary 3, for $\beta/2 > 0$) and α_f , we have two sub-horseshoes $*\Delta_f^{s,u}$ of $\Delta_f^{s,u}$, respectively, such that $*\Delta_f^{s,u} \cap \alpha_f = \emptyset$ and

$$HD(*K_f^s) \geq HD(K_f^s) - \beta/2 \quad \text{and} \quad HD(*K_f^u) \geq HD(K_f^u) - \beta/2.$$

Therefore,

$$HD(*K_f^s) \geq HD(K^s) - \beta \quad \text{and} \quad HD(*K_f^u) \geq HD(K^u) - \beta.$$

Lemma 21. *Let $f \in \mathcal{M} \cap \mathcal{C}$ and $*\Delta_f^{s,u}$ as above. If $(x, v) \in *\Delta_f^{s,u}$ and $t_0, t_1 \in (0, t_+(x, v))$ are critical points of $g(t) = f \circ \pi(\phi^t(x, v))$, then $\pi(\phi^{t_0}(x, v)) \neq \pi(\phi^{t_1}(x, v))$.*

Proof. Suppose the contrary, that is, $\gamma_v(t_0) = \gamma_v(t_1)$, where $\gamma_v(t) = \pi(\phi^t(x, v))$ is the geodesic such that $\gamma_v(0) = x$ and $\gamma_v'(0) = v$. Then, $g'(t) = df_{\gamma_t(v)}\gamma_v'(t) = \langle \nabla f(\gamma_v(t)), \gamma_v'(t) \rangle$, by construction of $*\Delta_f^{s,u}$, we have that $\nabla f(\gamma_v(t)) \neq 0$ for all $t \in (0, t_+(x, v))$, since

$g'(t_1) = g'(t_2) = 0$, then $\gamma'_v(t_1) = -\gamma'_v(t_2)$ or $\gamma_v(t_1) = \gamma_v(t_2)$. In the first case, we have a contradiction by the uniqueness of the geodesic. In the second case, we have that $\gamma_v(t)$ is a closed geodesic, since $t_0, t_1 \in (0, t_+(x, v))$, then (x, v) is a fixed point of \mathcal{R} , and this is a contradiction by Remark 14. \square

Proof of Lemma 19. Let us now consider the parametrizations given by Remark 12, $\varphi_i: U_{x_i} \times (-\epsilon, t_+(x_i) + \epsilon) \rightarrow \varphi_i(U_{x_i} \times (-\epsilon, t_+(x_i) + \epsilon))$. Let $f \in \mathcal{M} \cap \mathcal{C}$, put $F_i(x_1, x_2, x_3) = f \circ \varphi_i(x_1, x_2, x_3)$ the f in local coordinates. Then for $x \in {}_*\Delta_f^{s,u}$, the critical points of $F|_{\{x\} \times (-\epsilon, t_+(x) + \epsilon)}$ is finite. Thus the critical points of $F_i|_{\{z\} \times (-\epsilon, t_+(z) + \epsilon)}$ are locally graphs of a finite functions ψ_j , that is, are locally $\{(x_1, x_2, \psi_j(x_1, x_2)); 1 \leq j \leq k\}$, by Lemma 21, we can assume that

$$\pi(\{z = \psi_i(x_1, x_2)\}) \cap \pi(\{z = \psi_j(x_1, x_2)\}) = \emptyset \quad \text{for all } i \neq j. \quad (21)$$

Define a C^∞ -function, ξ_i close to the constant function 0 and equal to $-\gamma_i$ (γ_i is appropriately taken as in proof of Lemma 18) in neighborhood of $\pi(\{z = \psi_i(x_1, x_2)\})$ and 0 outside. The equation (21) implies that the functions $\xi_i \circ \pi$ have disjoint supports, therefore, the proof of the lemma follows similarly as the second part of the proof as Lemma 18. \square

Remark 15. By the proof of Lemma 19, the sub-horseshoe $\Delta_f^{s,u}$ has the property that

$$\{\pi(\phi^t(z)) : z \in \Delta_f^{s,u}\} \cap \{\text{Critical points of } f\} = \emptyset.$$

Corollary 4. The property of Lemma 19 is robust in the following sense: If X is a vector field C^1 -close to ϕ , then $\mathcal{C}_\phi = \mathcal{C}_X$ and for any $f \in \mathcal{C}_X$, holds that $\max(f \circ \pi)_X \in C^1(\Xi \cap R_f^{s,u}, \mathbb{R})$.

Consider now a surface \mathcal{S} defined by (cf. section 4.2.1)

$$\mathcal{S} = \{(x, v) \in SM : \exists t(x, v) \text{ such that } \gamma_v(t(x, v)) = x \text{ and } \{v, \gamma'_v(t(x, v))\} \text{ are LI}\}$$

Proposition 2. This surface satisfies the following properties:

1. If $z \notin \mathcal{S}$, then there is $\epsilon > 0$ such that $\phi^t(z) \notin \mathcal{S}$ for all $t \in (-\epsilon, \epsilon)$.
2. If $z \in \mathcal{S}$, then for all $\epsilon > 0$ small there exists a finite $|t_k| < \epsilon$ such that $\phi^{t_k}(z) \notin \mathcal{S}$.

Lemma 22. Keeping the same notation of Lemma 19, there exists $\mathcal{A}_\phi \subset \mathcal{C}_\phi$, $C^\infty(M, \mathbb{R})$ -dense and C^2 -open such that if $f_\phi := \max(f \circ \pi)_\phi|_{\Xi \cap R_f^{s,u}}$ for $f \in \mathcal{A}_\phi$ there is z_f a maximum point of f_ϕ in $\Delta_f^s \cup \Delta_f^u$, with

$$f_\phi(z_f) > f_\phi(z) \text{ for all } z \in (\pi^{-1}(\pi(z_f)) \setminus \{z_f\}) \cap (\Delta_f^s \cup \Delta_f^u).$$

Proof. The condition of Lemma is clearly C^2 -open, to prove the density, we have to prove simply that \mathcal{A}_ϕ is dense in \mathcal{C}_ϕ . In fact:

Let $f \in \mathcal{C}_\phi$ and $z_f = (x, v)$ maximum point of f_ϕ we can without loss of generality that $\pi^{-1}(\pi(z_f)) \cap \Xi$ and therefore $\#(\pi^{-1}(\pi(z_f)) \cap \Xi) < \infty$. Moreover, by Proposition 2, we can assume that $z_f \notin \mathcal{S}$. Suppose now that there is $z = (x, w) \in \pi^{-1}(\pi(z_f)) \cap (\Delta_f^s \cup \Delta_f^u)$ such that $f_\phi(z_f) = f_\phi(z)$, put \tilde{z}_f and \tilde{z} such that $f_\phi(z_f) = f \circ \pi(\tilde{z}_f) = f \circ \pi(\phi^{t(z_f)}(z_f))$ and $f_\phi(z) = f \circ \pi(\tilde{z}) = f \circ \pi(\phi^{t(z)}(z))$. Thus we have the following affirmations:

1. $\pi(\tilde{z}_f) \neq \pi(\tilde{z})$,
2. $\pi^{-1}(\pi(\tilde{z}_f)) \cap \{\phi^t(z) : t \in [t_-(z), t_+(z)]\} = \emptyset$.

Since $0 = \frac{d}{dt}(f \circ \pi(\phi^t(z_f)))|_{t=t(z_f)} = df_{\pi(\tilde{z}_f)}\gamma'_v(t(z_f))$ and $0 = \frac{d}{dt}(f \circ \pi(\phi^t(z)))|_{t=t(z)} = df_{\pi(\tilde{z})}\gamma'_w(t(z))$. Then, if $\pi(\tilde{z}_f) = \pi(\tilde{z})$, in any case, since Δ does not contains fixed points (cf. Remark 14) and $z_f \notin \mathcal{S}$, we have that set $\{\gamma'_v(t(z_f)), \gamma'_w(t(z))\} \subset T_{\pi(\tilde{z}_p)}M$ are linearly independent. This implies that $df_{\pi(\tilde{z}_f)} = 0$, and by Remark 15 this is a contradiction. This concludes the proof of 1.

Let us now prove 2, suppose by contradiction that $\phi^{t_1}(z) \in \pi^{-1}(\pi(\tilde{z}_f))$, then $f \circ \pi(\phi^{t_1}(z)) = f \circ \pi(\tilde{z}_f) = f \circ \pi(\tilde{z}) = f \circ \pi(\phi^{t(z)}(z))$, then by the uniqueness of the points of maximum $f \circ \pi(\phi^t(z))$, we have that $t_1 = t(z)$, i.e., $\tilde{z} = \phi^{t(z)}(z) = \phi^{t_1}(z) \in \pi^{-1}(\pi(\tilde{z}_f))$, which is a contradiction with 1.

The property 2 above, implies that for all $z \in (\pi^{-1}(\pi(z_f)) \setminus \{z_f\}) \cap (\Delta_f^s \cup \Delta_f^u)$ with $f_\phi(z_f) = f_\phi(z)$, there are neighborhoods V_z of z in Ξ and $U_{\pi(\tilde{z}_f)}$ of $\pi(\tilde{z}_f)$ such that

$$\left(\bigcup_{y \in V_z} \{\phi^t(y) : t \in [t_-(y), t_+(y)]\} \right) \cap \pi^{-1}(U_{\pi(\tilde{z}_f)}) = \emptyset. \quad (22)$$

Now, given $\epsilon > 0$ sufficiently small, let g be a C^∞ -function equal to ϵ in $\pi(\tilde{z}_f)$, $g \geq 0$ close to the constant 0 with maximum in $\pi(\tilde{z}_f)$ and 0 outside of $U_{\pi(\tilde{z}_f)}$, then z_f is a maximum point of $((f + g)_\phi)|_{\Xi \cap R_f^{s,u}}$. Moreover, if $z \in (\pi^{-1}(\pi(z_f)) \setminus \{z_f\}) \cap (\Delta_f^s \cup \Delta_f^u)$, then by equation (22), we have that $\phi^t(z) \notin \pi^{-1}(U_{\pi(\tilde{z}_f)})$, thus

$$\begin{aligned} (f + g) \circ \pi(\phi^t(z)) &= f(\pi(\phi^t(z))) + g(\pi(\phi^t(z))) \\ &< f(\pi(\tilde{z}_f)) + \epsilon = f(\pi(\tilde{z}_f)) + g(\pi(\tilde{z}_f)) = (f + g)_\phi(z_f). \end{aligned}$$

That is, $(f + g)_\phi(z) < (f + g)_\phi(z_f)$.

Moreover, the previous property together with the fact that $\pi^{-1}(\pi(z_f)) \cap \Xi$ implies that, there is a neighborhood U_{z_f} of z_f in Ξ such that

$$(f + g)_\phi(\tilde{z}) < (f + g)_\phi(x) \text{ for all } \tilde{z} \in (\pi^{-1}(\pi(x)) \setminus \{x\}) \cap (\Delta_f^s \cup \Delta_f^u) \text{ and } x \in U_{z_f}.$$

And this concludes the proof of Lemma, since $\Delta_{f+g}^{s,u}$ is close to $\Delta_f^{s,u}$. \square

Note that this Lemma also holds for vector fields C^1 -close to ϕ .

Corollary 5. *With the notation of Lemma 22, we have that, there is a neighborhood U_{z_f} of z_f such that*

$$f_\phi(x) > f_\phi(\tilde{x}) \text{ for all } x \in U_{z_f} \text{ and } \tilde{x} \in (\pi^{-1}(\pi(x)) \setminus \{x\}) \cap (\Delta_f^s \cup \Delta_f^u).$$

Proof. We can assume that $\pi^{-1}(\pi(z_f)) \cap \Xi$. Thus, by contradiction, suppose that there are $x_n \rightarrow z_f$ and $\tilde{x}_n \in \pi^{-1}(\pi(x_n)) \cap (\Delta_f^s \cup \Delta_f^u)$ such that $f_\phi(x_n) \leq f_\phi(\tilde{x}_n)$. By transversality, we have that $\tilde{x}_n \rightarrow w \in \pi^{-1}(\pi(z_f)) \setminus \{z_f\}$, therefore $f_\phi(z_f) = f_\phi(w)$ which contradicts the Lemma 22. \square

4.1.5 Proof of Theorem 1 and 2

In the proof of Theorems 1 and 2, we will use the following proposition found in ([MY10, pg. 21]).

Proposition 3. *Let Λ be a horseshoe and let $L \subset \Lambda$ an invariant proper subset of Λ . Then, for all $\epsilon > 0$, there is a sub-horseshoe $\tilde{\Lambda} \subset \Lambda$ such that $\tilde{\Lambda} \cap L = \emptyset$ and*

$$HD(\tilde{K}) \geq HD(K) - \epsilon,$$

where, K, \tilde{K} are of regular cantor set that describe the geometry transverse of the stable foliation $W^s(\Lambda), W^s(\tilde{\Lambda})$, respectively.

Proof of theorem 1 . Let $F \in \mathcal{B}_\phi$, $\Delta_F^{s,u}$ and $R_F^{s,u}$ as in Lemma 18 with

$$HD(K_F^{s,u}) \geq HD(K^{s,u}) - \beta.$$

Put $L = \Delta_F^s \cap \Delta_F^u \subset \Delta_F^s$ a \mathcal{R} -invariant set, then by Proposition 3 (applied to \mathcal{R}^{-1}), there is a sub-horseshoe ${}_1\Delta_F^s$ of Δ_F^s such that ${}_1\Delta_F^s \cap L = \emptyset$, that implies ${}_1\Delta_F^s \cap \Delta_F^u = \emptyset$. Moreover,

$$HD({}_1K_F^s) \geq HD(K_F^s) - \beta \geq HD(K^s) - 2\beta,$$

where ${}_1K_F^s$ is of regular cantor set that describe the geometry transverse of the unstable foliation $W^u({}_1\Delta_F^s)$. Define the sub-horseshoe Δ_F of Δ by $\Delta_F := {}_1\Delta_F^s \cup \Delta_F^u$, denote this by $\Delta_F := ({}_1\Delta_F^s, \Delta_F^u)$, put $R_F := R_F^s \cup R_F^u$ and consider the open and dense set

$$H_1(\mathcal{R}, \Delta_F) = \{f \in C^1(\Xi \cap R_F, \mathbb{R}) : \#M_f(\Delta_F) = 1 \text{ for } z \in M_f(\Delta_F), D\mathcal{R}_z(e_z^{s,u}) \neq 0\},$$

as in [MRn13, Theorem 1].

Let $f \in H_1(\mathcal{R}, \Delta_F)$, then there is a unique $z_f \in M_f(\Delta_F)$. Since ${}_1\Delta_F^s \cap \Delta_F^u = \emptyset$, we can suppose that $z_f \in {}_1\Delta_F^s$. Thus as in [MRn13, section 4], let ${}_1\tilde{\Delta}_F^s$ sub-horseshoe of ${}_1\Delta_F^s$ such that $HD({}_1\tilde{\Delta}_F^s) \sim HD({}_1\Delta_F^s)$ and $z_f \notin {}_1\tilde{\Delta}_F^s$, then put

$$\tilde{\Delta}_F = ({}_1\tilde{\Delta}_F^s, \Delta_F^u).$$

Moreover, since $HD({}_1\tilde{K}_F^s) \sim HD({}_1K_F^s)$, $HD(K^s) + HD(K^u) = HD(\Delta) \sim 2$ and β is small, then

$$HD({}_1\tilde{K}_F^s) + HD(K_F^u) > 1,$$

where ${}_1\tilde{K}_F^s$ is the regular Cantor set that describe the geometry transverse of the unstable foliation $W^u({}_1\tilde{\Delta}_F^s)$ (cf. subsection 5.4).

Hence, by [MY1] it is sufficient perturb ${}_1\Delta_F^s$ as in subsection 4.1.1, to obtain property V (cf. subsection 5.5). Let $\underline{w} \in \Omega$ such that $(\mathcal{R}^w, \tilde{\Delta}_F^w)$ has the property V , where $\tilde{\Delta}_F^w = ({}_1\tilde{\Delta}_w, \Delta_F^u)$ and ${}_1\tilde{\Delta}_w$ is the continuation of the hyperbolic set ${}_1\tilde{\Delta}_F^s$ for \mathcal{R}^w . Thus by Lemma 15, $\mathcal{R}^w = \mathcal{R}_{G_w}$, then $(\mathcal{R}_{G_w}, \tilde{\Delta}_F^w)$ has the property V so by [MRn13, Main Theorem] we have that

$$\text{int}M(f, \tilde{\Delta}_F^w) \neq \emptyset \text{ and } \text{int}L(f, \tilde{\Delta}_F^w) \neq \emptyset,$$

for any $f \in H_1(\mathcal{R}, \Delta_F)$. Now by Corollary 2 the function $\max F_{G_{\underline{w}}} |_{\Xi \cap (R_F^s \cup R_F^u)}$ is C^1 , using local coordinates as in Remark 13 respect to the field $G_{\underline{w}}$, we can find $g \in C^2(\Xi, \mathbb{R})$ such that

$$\max F_{G_{\underline{w}}} |_{\Xi \cap (R_F^s \cup R_F^u)}(x_1, x_2, x_3) + g(x_1, x_2) \in H_1(\mathcal{R}, \Delta_F). \quad (23)$$

Put $h(x_1, x_2, x_3) = F(x_1, x_2, x_3) + g(x_1, x_2)$, then $\max h_{G_{\underline{w}}} = \max F_{G_{\underline{w}}} + g \in H_1(\mathcal{R}, \Delta_F)$. Therefore, since $M(h, \tilde{\Delta}_F^w) = \left\{ \sup_{n \in \mathbb{Z}} h(\mathcal{R}_{G_{\underline{w}}}^n(x)) : x \in \tilde{\Delta}_F^w \right\} \subset M(h, G_{\underline{w}})$. So,

$$\text{int} M(h, G_{\underline{w}}) \neq \emptyset.$$

Analogously, $L(h, \tilde{\Delta}_F^w) \subset L(h, G_{\underline{w}})$, therefore $\text{int} L(h, G_{\underline{w}}) \neq \emptyset$. \square

Proof of theorem 2. Let $f \in \mathcal{A}_\phi$, put $F = f \circ \pi$. Let $\beta > 0$ be and consider $\Delta_f^{s,u}$ and $R_f^{s,u}$ (as in Lemma 19) with

$$HD(K_f^{s,u}) \geq HD(K^{s,u}) - \beta,$$

and $f_\phi := \max(f \circ \pi)_\phi |_{\Xi \cap (R_f^s \cup R_f^u)}$ is C^1 .

Similarly as in the proof of Theorem 1, we find the sub-horseshoe $\tilde{\Delta}_F^w$ of $\Delta_F := {}_1\Delta_f^s \cup \Delta_f^u$ with ${}_1\Delta_f^s \cap \Delta_f^u = \emptyset$ such that

$$\text{int} M(k, \tilde{\Delta}_F^w) \neq \emptyset \quad \text{and} \quad \text{int} L(k, \tilde{\Delta}_F^w) \neq \emptyset,$$

for any $k \in H_1(\mathcal{R}, \Delta_F)$. Also, by Corollary 4 the function $\max(f \circ \pi)_{G_{\underline{w}}} |_{\Xi \cap (R_f^s \cup R_f^u)}$ is C^1 .

Now, we want obtain (23) with $g = j \circ \pi$ with $j \in C^2(M, \mathbb{R})$. In fact:

Put $f_{\underline{w}} = \max(f \circ \pi)_{G_{\underline{w}}} |_{\Xi \cap (R_f^s \cup R_f^u)}$, and let $z_{\underline{w}} \in M_{f_{\underline{w}}}(\Delta_F) = \{z \in \Delta_F : f_{\underline{w}}(z) \geq f_{\underline{w}}(x) \forall x \in \Delta_F\}$ given by Lemma 22 (applied to the vector field $G_{\underline{w}}$) and let $U_{z_{\underline{w}}}$ be a neighborhood of $z_{\underline{w}}$ given by Corollary 5 such that

$$f_{\underline{w}}(x) > f_{\underline{w}}(\tilde{x}) \text{ for all } x \in U_{z_{\underline{w}}} \text{ and } \tilde{x} \in (\pi^{-1}(\pi(x)) \setminus \{x\}) \cap \Delta_F. \quad (24)$$

Now we have that there exists $g \in C^1(\Xi \cap (R_f^s \cup R_f^u), \mathbb{R})$ such that $g = 0$ outside of $U_{z_{\underline{w}}}$, and $f_{\underline{w}} + g \in H_1(\mathcal{R}, \Delta_F)$ with the maximum point $\tilde{z}_{\underline{w}}$ in $U_{z_{\underline{w}}}$, i.e., $\tilde{z}_{\underline{w}} \in M_{f_{\underline{w}}+g}(\Delta_F) \cap U_{z_{\underline{w}}}$. Without loss of generality, we can assume that $\pi|_{U_{z_{\underline{w}}}} : U_{z_{\underline{w}}} \rightarrow \pi(U_{z_{\underline{w}}})$ is a diffeomorphism, then define $j := g \circ (\pi|_{U_{z_{\underline{w}}}})^{-1} : \pi(U_{z_{\underline{w}}}) \rightarrow \mathbb{R}$ and put j equal to 0 outside of neighborhood of $\pi(U_{z_{\underline{w}}})$.

Claim: The function $f_{\underline{w}} + j \circ \pi \in H_1(\mathcal{R}, \Delta_F)$. In fact: We have to prove simply that $\tilde{z}_{\underline{w}}$ is the only maximum point of $f_{\underline{w}} + j \circ \pi$ in Δ_F . Since $j \circ \pi = g$ on $U_{z_{\underline{w}}}$, then $\tilde{z}_{\underline{w}} \in M_{f_{\underline{w}}+j \circ \pi}(\Delta_F) \cap U_{z_{\underline{w}}}$. Now, if $z \in (\pi^{-1}(\pi(U_{z_{\underline{w}}})) \setminus U_{z_{\underline{w}}}) \cap \Delta_F$, then there is $x \in U_{z_{\underline{w}}}$ such that $z \in \pi^{-1}(\pi(x))$, so $j \circ \pi(z) = j \circ \pi(x)$, moreover by inequality (24), $f_{\underline{w}}(x) > f_{\underline{w}}(z)$, this implies that

$$f_{\underline{w}}(\tilde{z}_{\underline{w}}) + j \circ \pi(\tilde{z}_{\underline{w}}) > f_{\underline{w}}(x) + j \circ \pi(x) > f_{\underline{w}}(z) + j \circ \pi(z).$$

Again, since $g = j \circ \pi = 0$ on $\Xi \setminus \pi^{-1}(\pi(U_{z_{\underline{w}}}))$, then the inequality above is true for $z \in \Delta_F \setminus \pi^{-1}(\pi(U_{z_{\underline{w}}}))$, this proves the claim.

The remainder of the proof follows similarly as the proof of theorem 1. \square

An important observation is that the vector field $G_{\underline{w}}$ is not necessarily a geodesic field for some Riemannian metric near the initial Riemannian metric.

4.2 The Interior of Spectrum for Geodesic Flow

In this section will prove a version of Theorem 1 and 2, where the vector field X is the geodesic field to some Riemannian metric near the initial Riemannian metric.

The main problem to obtain X in Theorem 1 and 2 as being a geodesic field is independence in the perturbation of the diffeomorphism \mathcal{R} , to obtain property V (cf. subsection 5.5), *i.e.*, in the proof of Theorem 1 we could perturb \mathcal{R} in each $R(\underline{a}) \in R_1$ without regard to the dynamics out.

If we want perturb \mathcal{R} to obtain property V and still be an application of first return of the geodesic flow for Riemannian metric near the initial Riemannian metric, we must keep in mind that upsetting a metric in a neighborhood of a point in the manifold M , then we affect the metric (of Sasaki in SM) in the points of the fiber of the neighborhood perturbed. *i.e.*, if the metric is perturbed in U , then the Sasaki metric is perturbed in $\pi^{-1}(U)$, where $\pi: SM \rightarrow M$ is the canonical projection $\pi(x, v) = x$. Therefore, if we want to perturb \mathcal{R} in $R(\underline{a}) \subset R_1$ as an application of first return of a geodesic flow, then such perturbation is not necessarily independent of $R(\underline{a})$.

Therefore, what we will do is obtain a sub-horseshoe $\bar{\Delta}$ of Δ with $HD(\bar{\Delta}) > 1$ and such that the perturbation of the metric, in a neighborhood of the image for $\pi(R^{1/2})$ for R in a suitable Markov partition of $\bar{\Delta}$ (cf. definition 3), induces a perturbation of \mathcal{R} as an application of a first return of the geodesic flow for a metric near, and that the perturbation be independent.

The next two sub-sections are aimed at finding such sub-Horseshoe.

4.2.1 The Set of Geodesics With Transversal Self-Intersection

Let $(x_0, v_0) \in SM$ such that the geodesic $\pi(\phi^t(x_0, v_0)) = \gamma_{v_0}(t)$ (with $\gamma_{v_0}(0) = x_0$ and $\gamma'_{v_0}(0) = v_0$) has a point of transversal self-intersection, that is, there is $t_0 \in \mathbb{R}$ such that $\phi^{t_0}(x_0, v_0) \in \pi^{-1}(x_0)$ and $\{v_0, \gamma'_{v_0}(t_0)\}$ is basis of $T_x M$.

Remark: We can see that since the Liouville measure is invariant by the geodesic flow and we are assuming that M has finite volume, then the set of geodesics with transverse self-intersection is not empty.

Let \mathcal{L} be a section transverse to flow and to the fiber $\pi^{-1}(x_0)$, define the following function

$$\begin{aligned} f: \mathcal{L} \times \mathbb{R} &\longrightarrow M \\ ((x, v), t) &\longmapsto \pi(\phi^t(x, v)) \end{aligned}$$

with $f((x_0, v_0), 0) = x_0$ and $f((x_0, v_0), t_0) = x_0$, let I_0, I_{t_0} are small intervals containing 0 and t_0 respectively. Denoted $f_0 = f|_{\mathcal{L} \times I_0}$ and $f_{t_0} = f|_{\mathcal{L} \times I_{t_0}}$. Let $\varphi: U_0 \subset T_{x_0}M \rightarrow U_{x_0}$ normal coordinates in x_0 , where U_{x_0} is neighborhood of x_0 , that is, let $\{e_1, e_2\}$ be orthonormal basis of $T_{x_0}M$ and $\varphi(x_1, x_2) = \varphi(x_1 e_1 + x_2 e_2) = \exp_{x_0}(x_1 e_1 + x_2 e_2)$.

We define

$$H: \mathcal{L} \times I_{t_0} \times I_0 \longrightarrow V_0 \subset T_{x_0}M$$

by $H((x, v), t, s) = (\varphi^{-1} \circ f_{t_0})((x, v), t) - (\varphi^{-1} \circ f_0)((x, v), s)$ satisfies $H((x_0, v_0), t_0, 0) = 0$.

Then,

$$\begin{aligned} \frac{\partial H}{\partial t}((x_0, v_0), t_0, 0) &= (D\varphi^{-1})_{f_{t_0}((x_0, v_0), t_0)} \left(\frac{\partial f_{t_0}}{\partial t}((x_0, v_0), t_0) \right) \\ &= (Dexp_{x_0}^{-1})_{x_0}(\gamma'_{v_0}(t_0)) = \gamma'_{v_0}(t_0) \end{aligned}$$

the last equality is due to the fact that $(Dexp_{x_0}^{-1})_{x_0} = Id: T_{x_0}M \rightarrow T_{x_0}M$ identity of $T_{x_0}M$. Also,

$$\begin{aligned} \frac{\partial H}{\partial s}((x_0, v_0), t_0, 0) &= -(D\varphi^{-1})_{f_0((x_0, v_0), 0)} \left(\frac{\partial f_0}{\partial s}((x_0, v_0), 0) \right) \\ &= -(Dexp_{x_0}^{-1})_{x_0}(\gamma'_{v_0}(0)) = -(Dexp_{x_0}^{-1})_{x_0}(v_0) \\ &= -v_0. \end{aligned}$$

Since $\{-v_0, \gamma'_{v_0}(t_0)\}$ are linearly independent, then, $\frac{\partial H}{\partial(t, s)}$ is an isomorphism. Therefore by the Implicit Function Theorem, there is an open $U_{\mathcal{L}}$ of (x_0, v_0) in \mathcal{L} and a diffeomorphism $\xi: U_{\mathcal{L}} \rightarrow V_{(t_0, 0)}$ with $V_{(t_0, 0)}$ open set containing $(t_0, 0)$ in $\mathbb{R} \times \mathbb{R}$ and $H((y, w), \xi(y, w)) = 0$. Without loss of generality we can assume that $V_{(t_0, 0)} = \tilde{I}_{t_0} \times \tilde{I}_0$ and

$$\xi(y, w) = (\xi_1(y, w), \xi_2(y, w)),$$

with ξ_1 close to t_0 and ξ_2 close to 0, this implies

$$exp_{x_0}^{-1}(\pi(\phi^{\xi_1(y, w)}(y, w))) = exp_{x_0}^{-1}(\pi(\phi^{\xi_2(y, w)}(y, w))),$$

so $\pi(\phi^{\xi_1(y, w)}(y, w)) = \pi(\phi^{\xi_2(y, w)}(y, w))$. Equivalently $\gamma_w(\xi_1(y, w)) = \gamma_w(\xi_2(y, w))$, where $\pi(\phi^t(y, w)) = \gamma_w(t)$ for any $(y, w) \in U_{\mathcal{L}}$. Consider the new section transverse to flow

$$\tilde{U}_{\mathcal{L}} = \{\phi^{\xi_2(y, w)}(y, w) : (y, w) \in U_{\mathcal{L}}\}.$$

Note that $\xi_1(x_0, v_0) = t_0$ and $\xi_2(x_0, v_0) = 0$, so $(x_0, v_0) \in \tilde{U}_{\mathcal{L}}$.

Let $(x, v) \in \tilde{U}_{\mathcal{L}}$, then there exists a unique $(y, w) \in U_{\mathcal{L}}$ such that $(x, v) = \phi^{\xi_2(y, w)}(y, w)$, so there exist a unique $\xi_1(y, w)$ such that

$$\begin{aligned} x = \pi(x, v) &= \pi(\phi^{\xi_2(y, w)}(y, w)) = \pi(\phi^{\xi_1(y, w)}(y, w)) \\ &= \pi(\phi^{\xi_1(y, w) - \xi_2(y, w)}(\phi^{\xi_2(y, w)}(y, w))) \\ &= \pi(\phi^{\eta(y, w)}(x, v)), \end{aligned}$$

where $\eta(y, w) = \xi_1(y, w) - \xi_2(y, w)$ is close to t_0 . This implies that for any $(x, v) \in \tilde{U}_{\mathcal{L}}$ there is $\eta(y, w)$ such that $\phi^{\eta(y, w)}(x, v) \in \pi^{-1}(x)$ and $\{v, \gamma'_v(\eta(y, w))\}$ are linearly independent.

From the above discussion we have that the set

$$\begin{aligned} \mathcal{S} &= \{(x, v) \in SM \mid \exists t(x, v) \text{ such that } \gamma_v(t(x, v)) = x \\ &\quad \text{and } \{v, \gamma'_v(t(x, v))\} \text{ is linearly independent} \} \end{aligned}$$

is a submanifold of SM of dimension 2.

Put $\mathcal{S}_n = \{(x, v) \in \mathcal{S} : |t(x, v)| < n\}$. Clearly, $\mathcal{S}_n \subset \mathcal{S}_{n+1}$. Given $(x, v) \in \mathcal{S}_n$, there is a neighborhood U of (x, v) in \mathcal{S} such that $U \subset \mathcal{S}_{n+1}$, therefore we can consider that \mathcal{S}_n is a surface, submanifold of SM .

4.2.2 Independent Perturbations of the Metric

In Lemma 10 was proven that there are GCS Σ_i such that

$$\Lambda \subset \bigcup_{i=1}^{m(l)} \phi^{(-2\gamma, 2\gamma)}(\text{int}(\Sigma_i)),$$

with $\Sigma_i \cap \Sigma_j = \emptyset$.

Then the hyperbolic set $\Delta = \bigcap_{n \in \mathbb{Z}} \mathcal{R}^n(\bigcup_i \Sigma_i)$, where \mathcal{R} is the Poincaré map (map of first return) of $\bigcup_i \Sigma_i := \Xi$, satisfies by Lemma 14 that $d := HD(\Delta)$ is very close to 2.

The Poincaré map is conservative, and so, by the Remark 11, given ϵ small, there is a Markov partition of Δ in rectangles whose sides have length of the order of ϵ . We will make a small abuse of language and call these rectangles with bounded distortion *squares of size ϵ* .

Definition 3. Let Δ_1, Δ_2 be two disjoint subhorseshoes of Δ . We say that Δ_1 has no interference on Δ_2 if there are Markov partitions \mathfrak{R}_1 of Δ_1 , and \mathfrak{R}_2 of Δ_2 , respectively, such that

$$T_{R_2} \cap \tau_{R_1^{1/2}} = \emptyset \text{ for any } R_1 \in \mathfrak{R}_1 \text{ and } R_2 \in \mathfrak{R}_2,$$

where $T_{R_2} = \{\phi^t(x) : x \in R_2 \text{ and } 0 \leq t \leq t_+(x)\}$, $\tau_{R_1^{1/2}} = \pi^{-1}(\pi(R_1^{1/2}))$ and $R_1^{1/2} = \phi^{\bar{t}_1/2}(R_1)$ with $\bar{t}_1 = \sup_{x \in R_1 \cap \Delta} t_+(x)$.

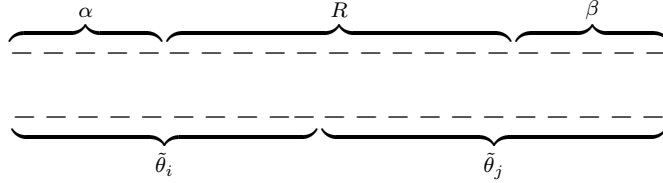
Lemma 23. Let $\Delta_1 \subset \Delta$ a sub-horseshoe with $0 < HD(\Delta_1) =: \lambda < \frac{1}{2}$, then there exists another sub-horseshoe Δ_2 of Δ with the following properties:

1. $HD(K_2^u)$ is very close to $HD(K^u)$, where K_2^u, K^u are the regular Cantor sets that describe the geometry transverse of the stable foliations $W^s(\Delta_2), W^s(\Delta)$, respectively.
2. $\Delta_1 \cap \Delta_2 = \emptyset$.
3. Δ_2 has no interference on Δ_1 .

Proof. Consider a Markov partition \mathfrak{R} of Δ in squares of size ϵ . Then \mathfrak{R} has approximately ϵ^{-d} squares, where $d = HD(\Delta)$. Consider analogously a set $\tilde{\mathfrak{R}}_{\Delta_1} \subset \mathfrak{R}$ of the order of $\epsilon^{-\lambda}$ squares of size ϵ forming a Markov partition of Δ_1 . Observe also that, given $p \in \Delta_1$ belonging to a cross section Σ_i , the projection by the flow ϕ^t of the fiber $\pi^{-1}(\pi(\phi^{t_+(p)/2}(p)))$ is a curve (or a finite union of curves), and so, as in the proof of Lemma 17, each square of $\tilde{\mathfrak{R}}_{\Delta_1}$ has interference on at most of the order of $\epsilon^{-d/2}$ (which is much smaller than ϵ^{-1}) squares of \mathfrak{R} . Thus, the squares of $\tilde{\mathfrak{R}}_{\Delta_1}$ have interference on at most $\epsilon^{-\lambda} \cdot \epsilon^{-1} \leq \epsilon^{-3/2}$ squares of \mathfrak{R} . We call $\mathfrak{X} \supset \tilde{\mathfrak{R}}_{\Delta_1}$ the set of squares which suffer interference of some square of $\tilde{\mathfrak{R}}_{\Delta_1}$. Therefore, we have $\tilde{N} := |\mathfrak{R} \setminus \mathfrak{X}| \geq \epsilon^{-d} - \epsilon^{-3/2}$ squares of \mathfrak{R} which do not suffer interference of any square of $\tilde{\mathfrak{R}}_{\Delta_1}$.

The maximal invariant set by \mathcal{R} of the union of these \tilde{N} remaining squares in $\mathfrak{R} \setminus \mathfrak{X}$, which will be the sub-horseshoe $\Delta_2 \subset \Delta$. By the construction above, this sub-horseshoe Δ_2 clearly satisfies the conditions 2 and 3 of lemma. In what follows, we will estimate the size of Δ_2 .

Let $\{\tilde{\theta}_1, \dots, \tilde{\theta}_{\tilde{N}}\}$ be the words associate with the remaining squares. They generate intervals of length of the order of ϵ^2 in $W^u(\Delta)$ (of the construction of the unstable regular Cantor set). Without loss of generality we can assume, as in Remark 10, that the transitions $\tilde{\theta}_i \tilde{\theta}_j$ are admissible for all $i, j \in \{1, \dots, \tilde{N}\}$. A transition $\tilde{\theta}_i \tilde{\theta}_j$ is said prohibited if there exists $R \in \mathfrak{X}$ (as a word) inside (a factor of) $\tilde{\theta}_i \tilde{\theta}_j$.



Since the product of the lengths of the intervals in $W^u(\Delta)$ generated by the words α and β is of the order of $\epsilon^4/\epsilon^2 = \epsilon^2$, the number of possibilities for the word $\#\{\alpha\beta\}$ is of the order of \tilde{N} . On the other hand, the size of each word $\tilde{\theta}_i$ (which gives an upper bound for the number of positions where the word R begins) is of the order of $\log \epsilon^{-d}$, which is of the order of $\log \tilde{N}$. Then each word R corresponding to a square in \mathfrak{X} prohibits $O(\tilde{N} \log \tilde{N})$ transitions. So we have in total $O(\epsilon^{-3/2} \tilde{N} \log \tilde{N})$ prohibited transitions $\tilde{\theta}_i \tilde{\theta}_j$. Since $d > 3/2$ and \tilde{N} is of the order of ϵ^{-d} , we have $\epsilon^{-3/2} \tilde{N} \log \tilde{N} = o(\tilde{N}^2)$.

This shows that the number of prohibited transitions is much smaller than the total number of transitions. So, consider the following matrix A for $i, j \in \{1, \dots, \tilde{N}\}$

$$a_{ij} = \begin{cases} 1 & \text{if } \tilde{\theta}_i \tilde{\theta}_j \text{ is not prohibited;} \\ 0 & \text{if } \tilde{\theta}_i \tilde{\theta}_j \text{ is prohibited for some } R \in \mathfrak{X} \end{cases}.$$

By the previous discussion we have (for ϵ small enough) $\#\{a_{ij} : a_{ij} = 1\} \geq \frac{99}{100} \tilde{N}^2$, so by Remark 10, the sub-horseshoe Δ_2 satisfies the condition 1 of the lemma. \square

Notice that the sub-horseshoe $\Delta_1 \subset \Delta$ in Lemma 23 with $0 < HD(\Delta_1) < \frac{1}{2}$ can be taken such that

$$HD(K_1^s) \sim 1/4, \quad (25)$$

where K_1^s is the regular stable Cantor set associated to Δ_1 . Now we create a family of sufficiently independent perturbations of \mathcal{R} in a neighbourhood of a suitable sub-horseshoe of Δ_1 .

Fix $n \in \mathbb{N}$ large and let \mathcal{S}_n be as in the subsection 4.2.1. Since the transversality condition is open and dense, then we can suppose that the GCS Σ_i are transverse to surface \mathcal{S}_n . This last implies that $\alpha_n := \bigcup_i \Sigma_i \cap \mathcal{S}_n$ is a finite family of smooth curves. Now by Lemma 17 applied to the family of curves α_n and the sub-horseshoe Δ_1 , we have that given $\tilde{\delta} > 0$ there is a sub-horseshoe Δ_0 of Δ_1 such that $\Delta_0 \cap \alpha = \emptyset$ for any $\alpha \in \alpha_n$ and

$$HD(K_0^s) \geq HD(K_1^s) - \tilde{\delta}, \quad (26)$$

where K_0^s, K_1^s are of regular Cantor sets that describe the geometry transverse of unstable foliation $W^u(\Delta_0), W^u(\Delta_1)$, respectively. Since the set of periodic points of \mathcal{R} of period smaller than n in Δ_1 is finite, we may also assume that Δ_0 does not contain any periodic point of period smaller than n .

Remark 16. *Given a positive integer m , we can perform the above construction for n large enough and choose a Markov partition \mathfrak{R}_0 of Δ_0 such that for each $R_a \in \mathfrak{R}_0$ there is a neighborhood U_a of R_a with the property $\phi^t(U_a) \cap \tau_{U_a^{1/2}} = \emptyset$ for $\inf_{x \in \overline{U_a}} t_0(x) < t \leq \sup_{x \in \overline{U_a}} \sum_{i=0}^m t_i(x)$ and for $-\sup_{x \in \overline{U_a}} \sum_{i=0}^m t_i(x) \leq t < 0$, where $t_i(x) = t_+(\mathcal{R}^i(x))$ and $\tau_{U_a^{1/2}} = \pi^{-1}(\pi(U_a^{1/2}))$. In particular, $\mathcal{R}^r(U_a) \cap \tau_{U_a^{1/2}} = \emptyset$ for $0 < r \leq m$.*

Let $\mathfrak{R}_0 = \{R_1, \dots, R_N\}$ be a Markov partition by squares of size $\epsilon^{1/2}$ of Δ_0 as in Remark 16. Note that since \mathcal{R} is conservative, then each square of this Markov partition \mathfrak{R}_0 , correspond to an interval of size of the order of ϵ in $W^s(\Delta_0)$ (of the constructions of the stable regular Cantor set of Δ_0): there is an iterate of the square which is a strip in the unstable direction, whose basis is this interval. We call $X := \{\theta_1, \dots, \theta_N\}$ the set of words associated to the intervals corresponding to the squares of \mathfrak{R}_0 in $W^s(\Delta_0)$. Without loss of generality (by considering, if necessary, a suitable sub-horseshoe with almost the same dimension) we can assume, as in Remark 10, that the transitions $\theta_i \theta_j$ are all admissible. We say that the word θ_i disturbs in the word θ_j if $i \neq j$ and $T_{R(\theta_j)} \cap \tau_{R(\theta_i)^{1/2}} \neq \emptyset$, where $R(\theta_i)$ is the square associated to the word θ_i . Define $P_{\theta_i} = \{\theta_j : \theta_i \text{ disturbs } \theta_j\} = \{\theta_{r_1(i)}, \dots, \theta_{r_{p_i}(i)}\}$. We have, as in the proof of Lemma 23, $|P_{\theta_i}| = O(N^{1/2})$.

Definition 4. *We say that θ_i prohibits the transition $\theta_j \theta_k$ if there exists a word $\theta_{r_1(i)} \in P_{\theta_i}$ inside (as a factor of) $\theta_j \theta_k$.*

For the next lemma we need some definitions and results given in the Appendix 5.2.

Lemma 24. *If $w \in T_x SM \setminus \{0\}$ is a vertical vector, then $w \notin E^{ss}(x) \oplus \phi(x)$ and $w \notin E^{uu}(x) \oplus \phi(x)$, where $E^{ss}(x), E^{uu}(x)$ are the stable and unstable space, respectively.*

Proof. Since SM is a hyperbolic set for ϕ , then $E^{ss}(x) \oplus E^{uu}(x) = \ker \alpha_x$ (cf. App. Lemma 30), where $\alpha_x : T_x SM \rightarrow \mathbb{R}$ is defined by $\alpha_x(\xi) = \langle d\pi_x(\xi), d\pi_x(\phi(x)) \rangle_{\pi(x)} = \langle d\pi_x(\xi), v \rangle_{\pi(x)}$. Suppose that $w = \alpha \xi^{ss} + \beta \phi(x)$ with $\xi^{ss} \in E^{ss}(x)$, then

$$0 = \langle d\pi_x(w), v \rangle = \langle \alpha d\pi_x(\xi^{ss}) + \beta d\pi_x(\phi(x)), v \rangle = \alpha \langle d\pi_x(\xi^{ss}), v \rangle + \beta \langle v, v \rangle = \beta.$$

Therefore, ξ^{ss} is a vertical vector, but

$$E^{ss}(x) = \{(J_s(0), J'_s(0)) \in H(x) \oplus V(x) : J_s \text{ is a stable Jacobi field}\},$$

thus $\xi^{ss} = 0$ and this is a contradiction (cf. Appendix 5.2.3). For $E^{uu}(x) \oplus \phi(x)$ the proof is analogous. \square

Lemma 25. Let $x \in \Delta$ and put $\tilde{x} = \phi^{\frac{t_+(x)}{2}}(x)$, then the surface

$$\eta_{\tilde{x}} := \bigcup_{t \in (-t_+(x), t_+(x))} \phi^t(\pi^{-1}(\pi(\tilde{x})))$$

intersects transversely Ξ , therefore $\eta_{\tilde{x}} \cap \Xi = \{\beta_1, \dots, \beta_l\}$ is a finite family of curves. Moreover, if $z \in \beta_i \cap \Delta \cap \Sigma$ for some $\Sigma \in \Xi$, then

$$\beta_i \pitchfork W^s(z, \Sigma) \quad \text{and} \quad \beta_i \pitchfork W^u(z, \Sigma).$$

Proof. For $X \in \pi^{-1}(\pi(\tilde{x}))$ and a non-zero vector $W \in T_X \pi^{-1}(\pi(\tilde{x}))$ (a tangent vector of $\pi^{-1}(\pi(\tilde{x}))$ in X), then the tangent space to $\eta_{\tilde{x}}$ in $\phi^t(X)$ is

$$T_{\phi^t(X)} \eta_{\tilde{x}} = \text{span}\{d\phi_X^t(W)\} \oplus \text{span}\{d\phi_X^t(\phi(X))\} = \text{span}\{d\phi_X^t(W)\} \oplus \text{span}\{\phi(\phi^t(X))\}.$$

Therefore, since $\phi \pitchfork \Xi$, then $\eta_{\tilde{x}} \pitchfork \Xi$, so $\eta_{\tilde{x}} \cap \Xi = \{\beta_1, \dots, \beta_l\}$.

Let $z \in \beta_i \cap \Delta \cap \Sigma$ for some $\Sigma \in \Xi$, then there is $X \in \pi^{-1}(\pi(\tilde{x}))$ such that $z = \phi^t(X)$, so

$$T_z \beta_i = T_z \eta_{\tilde{x}} \pitchfork T_z \Sigma = (\text{span}\{d\phi_X^t(W)\} \oplus \text{span}\{\phi(\phi^t(X))\}) \pitchfork T_z \Sigma,$$

where $W \in T_X \pi^{-1}(\pi(\tilde{x}))$ is a non-zero vector. Remember that $T_z W^i(z, \Sigma) = (E^i(z) \oplus \phi(z)) \cap T_z \Sigma$ for $i = ss, uu$, then since E^i is an invariant bundle for $d\phi^t$, $i = ss, uu$, it follows that $T_z W^i(z, \Sigma) = (d\phi_X^t(E^i(X)) \oplus \phi(\phi^t(X))) \pitchfork T_z \Sigma$, and by Lemma 24, $d\phi_X^t(W) \notin (d\phi_X^t(E^i(X)) \oplus \phi(\phi^t(X)))$ for $i = ss, uu$. The above concludes the proof of lemma. \square

Lemma 26. Given a constant $\delta \in (0, 1)$, there is a positive integer m such that, if Δ_0 is a sub-horseshoe of Δ_1 as in Remark 16 then, for any $i \leq N$, θ_i prohibits at most δN transitions of the type $\theta_i \theta_j$ or $\theta_j \theta_i$ (with $j \leq N$).

Proof. Let us first consider transitions of the type $\theta_i \theta_j$.

Write $\theta_i = \alpha\beta\gamma$ such that α is associated to an interval of size of the order of $\epsilon^{1/2}$ and β and γ are associated to intervals of sizes of the order of $\epsilon^{1/4}$ in $W^s(\Delta_0)$, and let $\alpha = s_1 s_2 \dots s_t$. Let $P_{\theta_i} = \{\theta_{r_1(i)}, \dots, \theta_{r_{p_i}(i)}\}$. If θ_i prohibits the transition $\theta_i \theta_j$ then there exists a word $\theta_{r_l(i)} \in P_{\theta_i}$ inside (as a factor of) $\theta_i \theta_j$.

Let us first show that the word $\theta_{r_l(i)}$ cannot begin too close from the beginning of θ_i itself. More precisely, if it begins by a letter s_k of α then we should have $k > m$. Indeed, if $k \leq m$ then the square R_i of the Markov partition \mathfrak{R}_0 corresponding to θ_i is such that $\mathcal{R}^{k-1}(R_i)$ intersects $R_{r_l(i)}$ (notice that $k > 1$ since, by definition, $r_l(i) \neq i$). Since, for some $x \in R_{r_l(i)}$, there is $0 \leq t \leq t_+(x)$ such that $\phi^t(R_{r_l(i)}) \cap \tau_{R_i^{1/2}} \neq \emptyset$, if we take $y \in R_i$ and $\tilde{t} = \sum_{i=0}^{k-2} t_i(y)$ (so that $\mathcal{R}^{k-1}(y) = \phi^{\tilde{t}}(y)$) then $\phi^{\tilde{t}+t}(U_i) \cap \tau_{U_i^{1/2}} \neq \emptyset$, a contradiction with Remark 16.

Consider now a word $\theta_{r_l(i)}$ beginning by the letter s_k of α (with $k > m$). Then, if $\tilde{\alpha}$ is the factor of θ_i beginning by the letter s_k of α (and also an initial factor of $\theta_{r_l(i)}$) associated to an interval of size of the order of $\epsilon^{1/2}$, the square associated to the word $\theta_{r_l(i)}$ belongs to a strip in the unstable direction corresponding to the interval (of size of the order of $\epsilon^{1/2}$) in $W^s(\Delta_0)$ associated to the word $\tilde{\alpha}$. The previous Lemma implies that there is a constant $\tilde{C} > 0$ (which depends on the transversality constants in the previous Lemma, but is independent of ϵ) such that θ_i disturbs at most \tilde{C} squares in this strip, so,

given k in this situation there are at most \tilde{C} possibilities for $\theta_{r_l(i)}$. For each such word, the largest part of it will be a factor of θ_i , and the remaining will be an initial factor $\hat{\alpha}$ of θ_j . Let m' be the minimum size of $\hat{\alpha}$. Then m' is of the order of m . There is a positive constant $\lambda < 1$ (a hyperbolicity constant for Δ) such that, for each $q \geq m'$, if the size of $\hat{\alpha}$ is q , then the number of words θ_j beginning by $\hat{\alpha}$ is at most $\lambda^q N$. Therefore, the number of prohibited transitions $\theta_i \theta_j$ in this situation is at most

$$N \cdot \sum_{q \geq m'} \tilde{C} \cdot \lambda^q = \frac{\tilde{C} N \lambda^{m'}}{1 - \lambda} < \frac{\delta N}{4}.$$

The next step is to consider a word $\theta_{r_l(i)}$ beginning by the letter s'_k of β . Then, if $\tilde{\beta}$ is the factor of θ_i beginning by the letter s'_k of β (and also an initial factor of $\theta_{r_l(i)}$) associated to an interval of size of the order of $\epsilon^{1/4}$, the square associated to the word $\theta_{r_l(i)}$ belongs to a strip in the unstable direction corresponding to the interval (of size of the order of $\epsilon^{1/4}$) in $W^s(\Delta_0)$ associated to the word $\tilde{\beta}$. Since the number of intervals of the construction of in $W^s(\Delta_0)$ whose sizes are of the order of $\epsilon^{1/2}$ contained in an interval of size $\epsilon^{1/4}$ is at most of the order of $N^{1/4}$ then, by the discussion of the previous step, the number of squares in this strip that are disturbed by θ_i is at most of the order of $N^{1/4}$. So, given k in this situation there are at most $N^{1/4}$ possibilities for $\theta_{r_l(i)}$. For each such word, a part of it will be a final factor of θ_i , and the remaining will be an initial factor $\hat{\alpha}$ of θ_j , which corresponds to an interval of size at most of the order of $\epsilon^{1/2}$. So, the number of words θ_j beginning by $\hat{\alpha}$ is at most of the order of $N^{1/2}$. Since the number of letters in β is of the order of $\log N$, the number of prohibited transitions $\theta_i \theta_j$ in this situation is $O(\log N \cdot N^{3/4}) = o(N)$.

The final step for estimating the number of prohibited transitions of the type $\theta_i \theta_j$ is considering the case when a word $\theta_{r_l(i)}$ begins by a letter of γ . In this case, a part of the word $\theta_{r_l(i)}$ will be a final factor of θ_i , and the remaining will be an initial factor $\hat{\alpha}$ of θ_j , which corresponds to an interval of size at most of the order of $\epsilon^{3/4}$. So, the number of words θ_j beginning by $\hat{\alpha}$ is at most of the order of $N^{1/4}$. Since $|P_{\theta_i}| = O(N^{1/2})$, the number of prohibited transitions $\theta_i \theta_j$ in this situation is $O(N^{3/4}) = o(N)$. Thus, the total number of prohibited transitions $\theta_i \theta_j$ is at most

$$\frac{\delta N}{4} + o(N) < \frac{\delta N}{2}.$$

The study of the case of transitions of the type $\theta_j \theta_i$ is analogous, and the total number of prohibited transitions in this case is also smaller than $\delta N/2$, which implies the result. We will only give some details of the argument corresponding to the first step: we show that the word $\theta_{r_l(i)}$ cannot end too close from the end of θ_i : it should end at least m letters before it. Indeed, if it ends k letters before the end of θ_i , and $k < m$ then the square $R_{r_l(i)}$ of the Markov partition \mathfrak{R}_0 corresponding to $\theta_{r_l(i)}$ is such that $\mathcal{R}^k(R_{r_l(i)})$ intersects R_i (notice that $k > 0$ since, by definition, $r_l(i) \neq i$). Since, for some $x \in R_{r_l(i)}$, there is $0 \leq t \leq t_+(x)$ such that $\phi^t(R_{r_l(i)}) \cap \tau_{R_i^{1/2}} \neq \emptyset$, if we take $y \in R_{r_l(i)}$ and $\tilde{t} = \sum_{i=0}^{k-1} t_i(y)$ (so that $\mathcal{R}^k(y) = \phi^{\tilde{t}}(y)$) then $\phi^{t-\tilde{t}}(U_i) \cap \tau_{U_i^{1/2}} \neq \emptyset$, a contradiction with Remark 16. \square

Now we will perform a probabilistic construction.
Fix a parameter α with $1/4 < \alpha < 1/2$.

Lemma 27. Let $f: \{1, \dots, \lfloor N^\alpha \rfloor\} \rightarrow X = \{\theta_1, \dots, \theta_N\}$ a random function (i.e., each value $f(i)$ is chosen randomly, with the uniform distribution, and independently from the other). Then f is injective with probability $1 - O_N(1)$.

Proof. The total number of functions f is $N^{\lfloor N^\alpha \rfloor}$. The number of injective functions among them is

$$\frac{N!}{(N - \lfloor N^\alpha \rfloor)!} = \prod_{j=0}^{\lfloor N^\alpha \rfloor - 1} (N - j).$$

So, the desired probability is

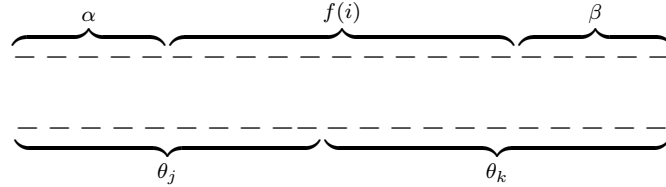
$$\frac{1}{N^{\lfloor N^\alpha \rfloor}} \prod_{j=0}^{\lfloor N^\alpha \rfloor - 1} (N - j) = \prod_{j=0}^{\lfloor N^\alpha \rfloor - 1} \left(1 - \frac{j}{N}\right) \geq 1 - \frac{\sum_{j=0}^{\lfloor N^\alpha \rfloor - 1} j}{N} \geq 1 - \frac{(N^\alpha)^2}{2N} = 1 - \frac{1}{2N^{1-2\alpha}} = 1 - O_N(1).$$

□

Given three indices $i, j, k \leq \lfloor N^\alpha \rfloor$ with $j \neq k$, we will estimate the probability that, given a random function $f: \{1, \dots, \lfloor N^\alpha \rfloor\} \rightarrow X$, $f(j)f(k)$ is prohibited by $f(i)$.

We have two cases:

- i) If $i \in \{j, k\}$, the above probability is at most δ , by Lemma 26.
- ii) If $i \notin \{j, k\}$, $i \in \{1, \dots, \lfloor N^\alpha \rfloor\}$, assume that $f(i)$ prohibits $\theta_j \theta_k$. Then the situation is as in the following diagram, where we have two representations of a same word:



The number of possibilities for the pair (α, β) is $O(N \log N)$ and so, since $|P_{f(i)}| = O(N^{1/2})$, we have

$$|\{\theta_j \theta_k : \theta_j \theta_k \text{ is prohibited by } f(i)\}| = O(N^{1/2} \cdot N \log N).$$

Therefore, the probability that the transition $f(j)f(k)$ is prohibited by $f(i)$ is

$$P(f(j)f(k) \text{ is prohibited the transition } f(i)) = \frac{O(N^{1/2} N \log N)}{N^2} = O(N^{-1/2} \log N). \quad (27)$$

Given such a function f and $j \neq k$, we say that the the transition $f(j)f(k)$ is *prohibited* if the transition $f(j)f(k)$ is prohibited by $f(i)$ for some $i \leq \lfloor N^\alpha \rfloor$.

The previous estimates imply that, since $\alpha < 1/2$, the expected number of prohibited transitions is at most

$$2\delta \lfloor N^\alpha \rfloor^2 + \lfloor N^\alpha \rfloor^3 \cdot O(N^{-1/2} \log N) = 2\delta \lfloor N^\alpha \rfloor^2 + O(N^{3\alpha-1/2}) < 3\delta \lfloor N^\alpha \rfloor^2.$$

Given such a function f , let $\theta_i = f(i)$.

It follows that the probability that the number of prohibited transitions $\theta_j\theta_k$ with $j \neq k$ is $\geq 4\delta\lfloor N^\alpha \rfloor^2$ is $\leq 3/4$.

Consider $A = (a_{ij})$ for $i, j \in \{1, \dots, \lfloor N^\alpha \rfloor\}$ the matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \theta_i\theta_j \text{ is not prohibited;} \\ 0 & \text{if } \theta_i\theta_j \text{ is prohibited by some } \theta_k \in \text{Im}f \end{cases}.$$

Then, with probability at least $1/4$, $|\{i, j \in \{1, \dots, \lfloor N^\alpha \rfloor\}; a_{ij} = 0\}|$ is at most $\lfloor N^\alpha \rfloor + 4\delta\lfloor N^\alpha \rfloor^2 < 5\delta\lfloor N^\alpha \rfloor^2$ (for N large). We assume now that f satisfies this condition and is injective, and that $\delta < 1/500$.

Define \overline{K} the regular Cantor set

$$\overline{K} := \{\theta_{i_1}\theta_{i_2} \cdots \theta_{i_n} \cdots \mid a_{i_k i_{k+1}} = 1, \forall k \geq 1\} \subset K_0^s.$$

By the previous we have $\#\{a_{ij} : a_{ij} = 1\} \geq \frac{99}{100}(\lfloor N^\alpha \rfloor)^2$, so by the Remark 10 we have

$$HD(\overline{K}) \sim \frac{\log \lfloor N^\alpha \rfloor}{-\log \epsilon} \sim \alpha HD(K_0^s).$$

Consider the sub-horseshoe of Δ_0 defined by

$$\Delta_3 := \bigcap_{n \in \mathbb{Z}} \mathcal{R}^n \left(\bigcup_{i, j \leq \lfloor N^\alpha \rfloor, a_{ij}=1} (R(\theta_i) \cap \mathcal{R}^{-1}(R(\theta_j))) \right),$$

where $R(\theta_i)$ is the square associated to the word θ_i .

Since the stable regular Cantor set K_3^s described by the transverse geometry of unstable foliation $W^u(\Delta_3)$ is equal to \overline{K} , by the above discussion we have

$$HD(K_3^s) \sim \alpha HD(K_0^s). \quad (28)$$

As $HD(\Delta) \sim 2$, then $HD(K^u) \sim 1$, then by Lemma 23 the sub-horseshoe Δ_2 satisfies that $\Delta_2 \cap \Delta_3 = \emptyset$ and $HD(K_2^u) \sim 1$. Also, combining the equations (25), (26) and (28) we have that $HD(K_3^s) \sim \alpha^{\frac{1}{4}}$. Therefore, since α can be taken equal to $\frac{1}{2} - 4\epsilon$, with small $\epsilon > 0$, then $HD(K_3^s) \sim \frac{1}{8} - \epsilon$, thus

$$HD(K_2^u) + HD(K_3^s) > 1.$$

From section 4.1, we described the family of perturbations given in [MY1 page 19-20], in which it is possible to obtain the property V (cf. section 5.5).

In [MY10] was proved that if \mathcal{R} is a diffeomorphism with two horseshoe Δ_2, Δ_3 disjoint, we can perturb \mathcal{R} in a Markov partition of Δ_3 without altering the dynamics in Δ_2 as in the section 4.1 and such that the new dynamics has a horseshoe with the property V (cf. section 5.5).

Let i, j be such that $a_{ij} = 1$, since $\theta_i \theta_j$ is not prohibited, then $\pi(T_{R(\theta_j)}) \cap \pi(R(\theta_i)^{1/2}) = \emptyset$. This implies that, if we perturb of metric g in $\pi(R(\theta_i)^{1/2})$, then this perturbation is independent, *i.e.*, the dynamic of \mathcal{R} in $R(\theta_j)$ for $j \neq i$ not changed. Also, the dynamic of \mathcal{R} in Markov partition of Δ_2 given Lemma 23 also does not change (cf. definition 3). We want to perturb of metric g in a neighborhood of $\pi(R(\theta_i)^{1/2})$. Since the diameter of θ_i is sufficiently small, we can assume that $\pi(R(\theta_i)^{1/2})$ is contained in a normal coordinate system, *i.e.*, there is a point $p \in \pi(R(\theta_i)^{1/2})$, an orthonormal basis $\{e_1, e_2\}$ of $T_p M$ and open set $\tilde{U} \subset T_p M$ such that the function

$$\varphi: \tilde{U} \rightarrow U_i^{1/2}$$

defined by $\varphi(x, y) = \exp_p(xe_1 + ye_2)$ is a diffeomorphism and $\pi(R(\theta_i)^{1/2}) \subset U_i^{1/2}$.

We want perturb the metric g and generate a perturbation in the diffeomorphism \mathcal{R} , such that satisfies the condition of the family of perturbation (cf. subsection 4.1.1).

Remark 17. *Given a metric \tilde{g} close to g , then geodesic flow of \tilde{g} still defines a Poincaré map $\mathcal{R}^{\tilde{g}}$ defined in the same cross-section where is defined \mathcal{R} .*

Let \tilde{g} be a Riemannian metric close to the metric g and such that the support of $\tilde{g} - g$ is contained in U_i and satisfies

$$S_{\tilde{g}}^1(x) \cap S_g^1(x) = \emptyset \quad \text{for all } x \in \pi(R(\theta_i)^{1/2}), \quad (29)$$

where $S_{\tilde{g}}^1(x) := \pi_{\tilde{g}}^{-1}(x) = \{v \in T_x M : \|v\|_{\tilde{g}} := \sqrt{\tilde{g}(v, v)} = 1\}$ and $S_g^1(x) = \pi_g^{-1}(x)$.

Using the notation of definition 3, consider V_i a neighborhood of $R(\theta_i)$ and put $\tilde{t}_i = \sup_{x \in R(\theta_i) \cap \Delta_3} t_+(x)$, then we defined the local Poincaré map $\mathcal{R}_{1/2}: V_i \rightarrow \phi^{\tilde{t}_i/2}(V_i)$ associated to the geodesic flow ϕ^t , and for \tilde{g} close to g , we can consider the local Poincaré $\mathcal{R}_{1/2}^{\tilde{g}}: V_i \rightarrow \phi^{\tilde{t}_i/2}(V_i)$ associate to geodesic flow $\phi_{\tilde{g}}^t$ of the metric \tilde{g} .

Lemma 28. *Let \tilde{g} be a Riemannian metric that satisfies the above, and the equation (29), then*

$$\mathcal{R}^{\tilde{g}}(W_{\mathcal{R}}^s(z)) \cap \mathcal{R}(W_{\mathcal{R}}^s(z)) = \emptyset \quad \text{for all } z \in R(\theta_i) \cap \Delta_3.$$

Proof. First we show that

$$R_{1/2}^{\tilde{g}}(W_{\mathcal{R}}^s(z)) \cap \mathcal{R}_{1/2}(W_{\mathcal{R}}^s(z)) = \emptyset \quad \text{for all } z \in R(\theta_i) \cap \Delta_3. \quad (30)$$

In otherwise, *i.e.*, there is $(x, X) \in \mathcal{R}_{1/2}^{\tilde{g}}(W_{\mathcal{R}}^s(z)) \cap \mathcal{R}_{1/2}(W_{\mathcal{R}}^s(z))$ and $(x, X) \in \pi_g^{-1}(U_i)$. Thus, there are $(y, Y), (p, P) \in W_{\mathcal{R}}^s(z)$ such that $\mathcal{R}_{1/2}(p, P) = \mathcal{R}_{1/2}^{\tilde{g}}(y, Y) = (x, X)$. Therefore, $1 = \|X\|_g = \|P\|_g = \left\| \mathcal{R}_{1/2}(p, P) \right\|_g = \left\| \mathcal{R}_{1/2}^{\tilde{g}}(y, Y) \right\|_g$ and since the support of $\tilde{g} - g$ is contained in U_i , then $\left\| \mathcal{R}_{1/2}^{\tilde{g}}(y, Y) \right\|_{\tilde{g}} = \left\| \mathcal{R}_{1/2}^{\tilde{g}}(y, Y) \right\|_g = 1$. This implies that $(x, X) \in S_{\tilde{g}}^1(x) \cap S_g^1(x)$ which is a contradiction with (29).

Since $\left\| \mathcal{R}_{1/2}^{\tilde{g}}(w, W) \right\|_{\tilde{g}} = 1$ for all $(w, W) \in W_{\mathcal{R}}^s(z)$, then by (29) $\mathcal{R}_{1/2}^{\tilde{g}}(W_{\mathcal{R}}^s(z)) \cap \pi_g^{-1}(U_i) = \emptyset$.

Therefore $\phi_g^t(\mathcal{R}_{\frac{1}{2}}^{\tilde{g}}(w, W)) = \phi^t(\mathcal{R}_{\frac{1}{2}}^{\tilde{g}}(w, W))$ for all $(w, W) \in W_{\mathcal{R}}^s(z)$, where ϕ_g^t and ϕ^t are the geodesic flow of the metrics \tilde{g} and g , respectively. So, the above and the equation (30) implies Lemma. \square

The next step is to exhibit the perturbations or families of perturbations of g that have the property (29).

Let $\alpha^w(x, y)$ be a continuous family of C^∞ real function with support contained \tilde{U} , C^∞ -close to constant function 0 and $\alpha^0(x, y) \equiv 0$. Moreover, if $w \neq 0$, then $\alpha^w(x, y) \neq 0$ for all $(x, y) \in \varphi^{-1}(\pi(R(\theta_i)^{1/2}))$. Thus, we can define a new family Riemannian metric g_i^w in local coordinates by setting:

a) $g_i^w = (1 + \alpha^w(x, y))g,$

b) $g_i^w = e^{\alpha^w(x, y)}g,$

c)

$$\begin{aligned} (g_i^w)_{00}(x, y) &= g_{00}(x, y) + \alpha^w(x, y) \\ (g_i^w)_{ij}(x, y) &= g_{ij}(x, y) \quad (i, j) \neq (0, 0). \end{aligned}$$

In any case a), b) or c), the family of metric g_i^w satisfies the property (29) and therefore satisfies the Lemma 28.

We denote by \mathcal{R}_i^w the Poincaré map given by g_i^w (cf. Remark 17). Define the following application Φ_i^w on $R(\theta_i)$ by

$$\Phi_i^w(x, v) := \mathcal{R}^{-1} \circ \mathcal{R}_i^w(x, v) \quad \text{for } (x, v) \in R(\theta_i).$$

Corollary 6. *If $z \in \Delta_3 \cap R(\theta_i)$ e $w \neq 0$, then $\Phi_i^w(W_{\mathcal{R}}^s(z)) \cap W_{\mathcal{R}}^s(z) = \emptyset$.*

Proof. By Lemma 28, we have $\mathcal{R}_i^w(W_{\mathcal{R}}^s(z)) \cap \mathcal{R}(W_{\mathcal{R}}^s(z)) = \emptyset$, so $\Phi_i^w(W_{\mathcal{R}}^s(z)) \cap W_{\mathcal{R}}^s(z) = \emptyset$. \square

We define the new metric g_w on M close to the metric g by

$$g_w = \begin{cases} g_i^w & \text{if } x \in U_i; \\ g & \text{otherwise} \end{cases}.$$

Put $\Phi^w(x, v) := \Phi_i^w(x, v)$ if $(x, v) \in R(\theta_i)$. This Lemma implies that the perturbation of \mathcal{R} , given by $\mathcal{R}^w := \mathcal{R} \circ \Phi^w$ satisfies the condition on the family of perturbation to get the property V (cf. subsection 4.1.1 and section 5.5).

Consider a Riemannian metric g_w , then put $S^w M = \{(x, v) \in TM : \|v\|_{g_w} = 1\}$, the unitary tangent bundle associated to metric g_w . Then, there is a diffeomorphism $\mathcal{S}^w : S^w M \rightarrow SM$ defined by $\mathcal{S}^w(x, v) = (x, \frac{v}{\|v\|})$. If g^w, g are C^k metric, then \mathcal{S}^w is C^k . Moreover, g^w is C^k close to g , then \mathcal{S}^w is close to the identity. In the following sense: Let $(x, v) \in S^w M$ and $\psi_w : U \subset \mathbb{R}^3 \rightarrow S^w M$ a chart of $S^w M$, with $\psi_w(0) = (x, v)$ and

$\psi: V \subset \mathbb{R}^3 \rightarrow SM$ a chart of SM such that $\psi(0) = \mathcal{S}^w(x, v)$, then $\psi^{-1} \circ \mathcal{S}^w \circ \psi: U \rightarrow V$ is C^k diffeomorphism close to the identity of \mathbb{R}^3 .

Let ϕ_w^t be the geodesic flow of the metric g_w and ϕ_w the vector field that generates the geodesic flow ϕ_w^t . Define the vector field on SM by $J_w(x, v) = D\mathcal{S}^w_{(\mathcal{S}^w)^{-1}(x, v)}\phi_w((\mathcal{S}^w)^{-1}(x, v))$, is easy to see that the integral curve of J_w in a point $(x, v) \in SM$ is given by $\beta(t) = \mathcal{S}^w \circ \phi_w^t \circ (\mathcal{S}^w)^{-1}(x, v)$, that is $J_w(\beta(t)) = \beta'(t)$ and the flow of J_w is $J_w^t(x, v) = \beta(t)$.

Note that, since g_w is close to g , then \mathcal{S}^w is close to the identity, therefore the vector field J_w is close to ϕ , the vector field that generates the geodesic flow of the metric g . Thus, we can define the Poincaré map $\mathcal{R}^{J_w}: \Xi \rightarrow \Xi$ associated to J_w .

Now let us relate the diffeomorphism \mathcal{R}^{J_w} and \mathcal{R}^w .

Lemma 29. *The diffeomorphisms \mathcal{R}^{J_w} and \mathcal{R}^w are equal.*

Proof. Let $(x, v) \in SM$, then by definition

$$\phi_w^t((\mathcal{S}^w)^{-1}(x, v)) = \phi_w^t\left(x, \frac{v}{\|v\|_w}\right) = \left(\gamma_{\frac{v}{\|v\|_w}}(t), \gamma'_{\frac{v}{\|v\|_w}}(t)\right),$$

where $\gamma_{\frac{v}{\|v\|_w}}(t) = \exp_x \frac{v}{\|v\|_w} t = \gamma_v(\frac{t}{\|v\|_w})$ (with the metric g_w).

So, $\phi_w^t((\mathcal{S}^w)^{-1}(x, v)) = \left(\gamma_v(\frac{t}{\|v\|_w}), \frac{1}{\|v\|_w} \gamma'_v(\frac{t}{\|v\|_w})\right)$, but $\|\gamma'_v(t)\|_w = \|v\|_w$ for all t . Therefore,

$$\phi_w^t((\mathcal{S}^w)^{-1}(x, v)) = \left(\gamma_v\left(\frac{t}{\|v\|_w}\right), \frac{\gamma'_v\left(\frac{t}{\|v\|_w}\right)}{\left\|\gamma'_v\left(\frac{t}{\|v\|_w}\right)\right\|_w}\right) = \mathcal{K}^w\left(\gamma_v\left(\frac{t}{\|v\|_w}\right), \gamma'_v\left(\frac{t}{\|v\|_w}\right)\right), \quad (31)$$

where $\mathcal{K}^w: TM \setminus \{0\} \rightarrow S^w M$ define by $\mathcal{K}^w(y, Y) = (y, \frac{Y}{\|Y\|_w})$, where TM is a tangent bundle of M and $TM \setminus \{0\} = \bigcup_{x \in M} (T_x M \setminus \{0\})$. Moreover, $\mathcal{K}^w|_{SM} = (\mathcal{S}^w)^{-1}$.

Let $\Gamma_w^t: TM \rightarrow TM$, the geodesic flow defined by the metric g_w in TM , then the equation (31) implies that

$$J_w^t(x, v) = \mathcal{S}^w \circ \phi_w^t \circ (\mathcal{S}^w)^{-1}(x, v) = \mathcal{S}^w \circ \mathcal{K}^w(\Gamma_w^{t/\|v\|_w}(x, v)). \quad (32)$$

Now by definition of \mathcal{R}^w , we have that there is $t(x, v)$ (the first time) such that

$$\mathcal{R}^w(x, v) = \Gamma_w^{t(x, v)/\|v\|_w}(x, v) \in \Xi \subset SM.$$

Therefore, $\mathcal{K}^w(\Gamma_w^{t(x, v)/\|v\|_w}(x, v)) = (\mathcal{S}^w)^{-1}(\Gamma_w^{t(x, v)/\|v\|_w}(x, v))$, so by equation (32)

$$J_w^{t(x, v)}(x, v) = \Gamma_w^{t(x, v)/\|v\|_w}(x, v).$$

Since J_w is close to ϕ , then the equation above implies that $\mathcal{R}^{J_w}(x, v) = J_w^{t(x, v)}(x, v) = \mathcal{R}^w(x, v)$. \square

4.2.3 Proof of Main Theorem 1, 2

Proof of Main Theorem 1. Let us apply Lemma 18 and Corollary 2 to Δ_2 and Δ_3 . We find a set dense $\mathcal{B}_{J_w} \in C^\infty(SM, \mathbb{R})$, C^2 -open, such that given $\epsilon > 0$, then for any $F \in \mathcal{B}_{J_w}$, there are sub-horseshoes Δ_F^s of Δ_3 and Δ_F^u of Δ_2 with $HD(K_F^s) \geq HD(K_3^s) - \epsilon$ and $HD(K_F^u) \geq HD(K_2^u) - \epsilon$ (as in Lemma 17). Also, there are Markov partitions $R_F^{s,u}$ of $\Delta_F^{s,u}$, respectively, such that the function $\max F_{J_w}|_{\Xi \cap R_F^{s,u}} \in C^1(\Xi \cap R_F^{s,u}, \mathbb{R})$. Since $\Delta_2 \cap \Delta_3 = \emptyset$, the above implies that $\Delta_F^s \cap \Delta_F^u = \emptyset$ and

$$HD(\Delta_F^s) + HD(\Delta_F^u) > 1.$$

Hence, by [MY1] it is sufficient perturb Δ_F^s as in subsection 4.1.1, to obtain property V (cf. section 5.5). By Lemma 28 there is w small such that $(\mathcal{R}^w, \Delta_F^w)$ has the property V , where $\Delta_F^w = ((\Delta_F^s)^w, \Delta_F^u)$ and $(\Delta_F^s)^w$ is the continuation of the hyperbolic set Δ_F^s for \mathcal{R}^w . The Lemma 29 implies that $\mathcal{R}^w = \mathcal{R}_{J_w}$, then $(\mathcal{R}^J, \Delta_F^w)$ has the property V .

Continuing analogously as in the proof of Theorem 1, we conclude that there is an open and dense set $\mathcal{V} \subset C^2(SM, \mathbb{R})$, such that

$$\text{int}M(h, J_w) \neq \emptyset \quad \text{and} \quad \text{int}L(h, J_w) \neq \emptyset, \quad \text{for all } h \in \mathcal{V}.$$

Consider now, $k = h \circ \mathcal{S}^w \in C^2(S^w M, \mathbb{R})$, for $h \in \mathcal{V}$, then for $(y, w) = (\mathcal{S}^w)^{-1}(x, v) \in S^w M$, we have that $k(\phi_w^t(y, w)) = h(\mathcal{S}^w(\phi^t((\mathcal{S}^w)^{-1}(x, v)))) = h(J_w^t(x, v))$. Thus,

$$\text{int}M(k, \phi_w) \neq \emptyset \quad \text{and} \quad \text{int}L(k, \phi_w) \neq \emptyset, \quad \text{for all } k \in \mathcal{H},$$

where $\mathcal{H} = \{h \circ \mathcal{S}^w : h \in \mathcal{V}\}$ is dense an open set of $C^2(S^w M, \mathbb{R})$. □

Proof of Main Theorem 2. Joining the proof of Main Theorem 1 and the proof of Theorem 2, we have that there is an open and dense set \mathcal{I} , such that

$$\text{int}M(h \circ \pi, J_w) \neq \emptyset \quad \text{and} \quad \text{int}L(h \circ \pi, J_w) \neq \emptyset \quad \text{for all } h \in \mathcal{I}.$$

For $(y, w) = (\mathcal{S}^w)^{-1}(x, v) \in S^w M$, as $\pi|_{S^w M} = \pi|_{SM}$, and for $h \in \mathcal{I}$

$$h \circ \pi(\phi_w^t(y, w)) = h \circ \pi(\mathcal{S}^w(\phi_w^t((\mathcal{S}^w)^{-1}(x, v)))) = h \circ \pi(J_w^t(x, v)).$$

Therefore,

$$\text{int}M(h \circ \pi, \phi_w) \neq \emptyset \quad \text{and} \quad \text{int}L(h \circ \pi, \phi_w) \neq \emptyset, \quad \text{for all } h \in \mathcal{I}.$$

□

5 Appendix

5.1 Hyperbolic Flows

Let M be a smooth manifold, $\varphi : \mathbb{R} \times M \rightarrow M$ a C^r flow, $\Lambda \subset M$ a φ^t -invariant set. The set Λ is said to be *hyperbolic set for the flow φ^t* if there exist a Riemannian metric on an open neighborhood U of Λ , such that there is continuous splitting $T_\Lambda M = E^{ss} \oplus \varphi \oplus E^{uu}$ invariant under the derivative of the flow $D\phi$ on $T_\Lambda(SM)$, such that φ is the subbundle spanned by the direction of flow, $D\varphi$ exponentially expands E^{uu} , and $D\varphi$ exponentially contracts E^{ss} , that is, there are constants $K > 0$ and $\lambda > 0$ such that

$$|D\varphi^t(v)| \geq K^{-1}e^{\lambda t}|v| \quad \text{if } v \in E^{uu} \quad \text{and } t \geq 0,$$

$$|D\varphi^t(v)| \leq Ke^{-\lambda t}|v| \quad \text{if } v \in E^{ss} \quad \text{and } t \geq 0.$$

By the Stable and Unstable Manifold Theorem [KH95] it follows that there is $\epsilon > 0$ such that for every $x \in \Lambda$ the set

$$W_\epsilon^{ss}(x) = \{y : d(\varphi^t(x), \varphi^t(y)) \leq \epsilon \text{ and } d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t \rightarrow +\infty]{} 0\}$$

and

$$W_\epsilon^{uu}(x) = \{y : d(\varphi^t(x), \varphi^t(y)) \leq \epsilon \text{ and } d(\varphi^t(x), \varphi^t(y)) \xrightarrow[t \rightarrow -\infty]{} 0\}$$

are invariant C^r -manifolds tangent to E_x^{ss} and E_x^{uu} respectively at x . Then we call $W_\epsilon^{ss}(x)$ the local *strong-stable manifold* and $W_\epsilon^{uu}(x)$ the local *strong-unstable manifold*, sometimes denoted by $W_{loc}^{ss}(x)$ and $W_{loc}^{uu}(x)$, respectively. Here d is the distance on M induced by the Riemannian metric. Moreover, the manifolds $W_\epsilon^{ss}(x)$ and $W_\epsilon^{uu}(x)$ varies continuously with x . Also, if $x \in \Lambda$ one has that

$$W^s(x) = \bigcup_{t \geq 0} \varphi^{-t}(W_\epsilon^{ss}(\varphi^t(x))) \quad \text{and} \quad W^u(x) = \bigcup_{t \leq 0} \varphi^{-t}W_\epsilon^{uu}(\varphi^t(x))$$

are C^r invariant manifolds immerse in M , called of *strong-stable manifold* and *strong-unstable manifold* of x , respectively. Finally, the sets

$$W^{cs}(x) = \bigcup_{t \in \mathbb{R}} W^s(\varphi^t(x)) \quad \text{and} \quad W^{cu}(x) = \bigcup_{t \in \mathbb{R}} W^u(\varphi^t(x))$$

are invariant C^r manifolds tangent to $E_x^{ss} \oplus \varphi(x)$ and $E^{uu} \oplus \varphi(x)$, respectively.

5.2 Geometry of TM and SM

The following two subsections can be found in [Pat99]:

5.2.1 Vertical and horizontal subbundles

Let $\pi: TM \rightarrow M$ the canonical projection, *i.e.*, if $y = (x, v) \in TM$, then $\pi(y) = x$. The vertical subbundle is defined by $V(y) = \ker(d\pi_y)$. The connection map

$$K: TTM \rightarrow TM,$$

is defined as follows. Let $\xi \in T_y TM$ and $z: (-\epsilon, \epsilon) \rightarrow TM$ be a curve adapted to ξ , that is, with initial conditions as follows:

$$\begin{cases} z(0) = y; \\ z'(0) = \xi. \end{cases}$$

Such a curve gives rise to a curve $\alpha: (-\epsilon, \epsilon) \rightarrow M$, $\alpha := \pi \circ z$, and a vector field Z along α , equivalently, $z(t) = (\alpha(t), Z(t))$. Define

$$K_y(\xi) = (\nabla_\alpha Z)(0),$$

where ∇ is the Levi-Civita connection, and TTM is the tangent bundle of TM .

The horizontal subbundle is the subbundle of TTM whose fibre at y is given by $H(y) = \ker K_y$.

For the vertical and the horizontal subbundle holds that

$$T_y TM = H(y) \oplus V(y),$$

and that the map $j_y: T_y TM \rightarrow T_x M \times T_x M$ given by

$$j_y(\xi) = (d\pi_y(\xi), K_y(\xi)),$$

is a linear isomorphism.

We write $\xi = (\xi_h, \xi_v)$ we mean that we identify ξ with $j_y(\xi)$, where $\xi_h = d\pi_y(\xi)$ and $\xi_v = K_y(\xi)$.

Using the decomposition $T_y TM = H(y) \oplus V(y)$, we can define in a natural way a Riemannian metric on TM that makes $H(y)$ and $V(y)$ orthogonal. This metric is called the Sasaki metric and is given by

$$\langle \langle \xi, \eta \rangle \rangle_y = \langle d\pi_y(\xi), d\pi_y(\eta) \rangle_{\pi(y)} + \langle K_y(\xi), K_y(\eta) \rangle_{\pi(y)}.$$

The one-form α of TM defined by

$$\alpha_y(\xi) = \langle \langle \xi, \phi(y) \rangle \rangle = \langle d\pi_y(\xi), v \rangle_x,$$

such that α restricted to SM (the unit tangent bundle) it becomes a contact form whose the characteristic flow is the geodesic flow restrict to SM .

5.2.2 Jacobi fields and the differential of the geodesic flow

In this section we shall describe an isomorphism between the tangent space $T_y TM$ and the Jacobi fields along the geodesic γ_y . Using the decomposition of $T_y TM$ in vertical and horizontal subspaces, we shall give a very simple expression for the differential of the geodesic flow in terms of Jacobi fields. Recall that a Jacobi vector field along the geodesic

γ_y is a vector field along γ_y that is obtained as the variational vector field of a variation of y through geodesics. It is well known that J is a Jacobi vector field along γ_y if and only if it satisfies the *Jacobi equation*

$$J'' + R(\gamma'_y, J)\gamma'_y = 0,$$

where R is the Riemann curvature tensor of M and $'$ denote covariant derivatives along γ_y .

Let $\xi \in T_y TM$ and $z: (-\epsilon, \epsilon) \rightarrow TM$ be an adapted curve to ξ . Then the map $(s, t) \rightarrow \pi \circ \phi^t(z(s))$ gives rise to a variation of $\gamma_y = \pi \circ \phi^t(y)$. The curves $t \rightarrow \pi \circ \phi^t(z(s))$ are geodesics and therefore the corresponding variational vector field $J_\xi(t) = \frac{\partial}{\partial s}|_{s=0} \pi \circ \phi^t(z(s))$ is a Jacobi vector field with initial conditions given by

$$\begin{cases} z(0) = y; \\ z'(0) = \xi. \end{cases}$$

$$\begin{cases} J_\xi(0) = \frac{\partial}{\partial s}|_{s=0} \pi \circ \phi^t(z(s))|_{t=0} = d\pi_y(\xi); \\ J'_\xi(0) = \frac{\partial}{\partial t}|_{t=0} \frac{\partial}{\partial s}|_{s=0} \pi \circ \phi^t(z(s)) \\ \quad = \frac{\partial}{\partial s}|_{s=0} \frac{\partial}{\partial t}|_{t=0} \pi \circ \phi^t(z(s)) = \frac{\partial}{\partial s}|_{s=0} Z(s) = K_y(\xi). \end{cases}$$

Using the above we can describes the differential of the geodesic flow in terms of Jacobi fields and the splitting of $T_y TM$ into horizontal and vertical subbundles. In fact, holds that

Claim: Given $y \in TM$, $\xi \in T_y TM$ and $t \in \mathbb{R}$, we have

$$d\phi_y^t(\xi) = (J_\xi(t), J'_\xi(t)).$$

The following Lemma can be found in (cf. [Pat99, pag. 42]) for compact case, but in non-compact case the proof still holds with some adaptations.

Lemma 30. *Let $X \subset SM$ be a hyperbolic set. Then for any $y \in X$*

$$E^{ss}(y) \oplus E^{uu}(y) = \ker \alpha_y.$$

Proof. Let us show that $E^{ss}(y) \subset \ker \alpha_y$ the proof for $E^{uu}(y)$ is analogous. Since ϕ^t preserves the contact form α , then given $\eta \in E^{ss}(y)$, we have

$$\begin{aligned} \alpha_y(\eta) &= \alpha_{\phi^t(y)}(d\phi_y^t(\eta)) \\ &= \langle d\pi_{\phi^t(y)}(d\phi_y^t(\eta)), \gamma'_y(t) \rangle = \langle d(\pi \circ \phi^t)_y(\eta), \gamma'_y(t) \rangle \\ &= \langle J_\eta(t), \gamma'_y(t) \rangle, \end{aligned}$$

since $\|d\phi_y^t(\eta)\|^2 = \|J_\eta(t)\|^2 + \|J'_\eta(t)\|^2$ and $\|d\phi_y^t(\eta)\| \rightarrow 0$ when $t \rightarrow \infty$, then $\|J_\eta(t)\| \rightarrow 0$ when $t \rightarrow \infty$. Also, $|\alpha_y(\eta)| \leq \|J_\eta(t)\|$, so we have that $\alpha_y(\eta) = 0$ showing that $E^{ss}(y) \subset \ker \alpha_y$; therefore since $E^{ss}(y) \oplus E^{uu}(y)$ and $\ker \alpha_y$ have the same dimensions, thus $E^{ss}(y) \oplus E^{uu}(y) = \ker \alpha_y$. \square

5.2.3 Stable and Unstable Jacobi Fields

Let $y = (x, v)$ and w orthogonal to v and let $J_w^T(t)$ be the unique Jacobi field on $\gamma_v(t)$ such that

$$J_w^T(0) = w \quad \text{and} \quad J_w^T(T) = 0.$$

The limit $J_w^s(t) := \lim_{T \rightarrow \infty} J_w^T(t)$ exists and is a Jacobi vector field on $\gamma_v(t)$ (cf. [Ebe73]).

Clearly $J_w^s(0) = w$ and $J_w^s(t) \neq 0$ for all $t > 0$. We call $J_w^s(t)$ a *stable Jacobi field*.

The *unstable Jacobi field* $J_w^u(t)$ along $\gamma_v(t)$ are got by considering the limits as $T \rightarrow -\infty$,

$$J_w^u(t) := \lim_{T \rightarrow -\infty} J_w^T(t).$$

The subspaces (using the identification $T_y SM = H(y) \oplus V(y)$)

$$\begin{aligned} E^{ss}(\phi^t(y)) &= \{(J(t), J'(t)) \in T_{\phi^t(y)} SM \mid J \text{ is a stable Jacobi field}\} \\ E^{uu}(\phi^t(y)) &= \{(J(t), J'(t)) \in T_{\phi^t(y)} SM \mid J \text{ is a unstable Jacobi field}\} \end{aligned}$$

are called the Green subbundles on γ_y , which are also the stable and unstable subbundles of the definition of hyperbolicity of the geodesic flow on SM (cf. [Ebe73]).

5.3 Regular Cantor Sets

Let \mathbb{A} be a finite alphabet, \mathbb{B} a subset of \mathbb{A}^2 , and $\Sigma_{\mathbb{B}}$ the subshift of finite type of $\mathbb{A}^{\mathbb{Z}}$ with allowed transitions \mathbb{B} . We will always assume that $\Sigma_{\mathbb{B}}$ is topologically mixing, and that every letter in \mathbb{A} occurs in $\Sigma_{\mathbb{B}}$.

An *expansive map of type* $\Sigma_{\mathbb{B}}$ is a map g with the following properties:

- (i) the domain of g is a disjoint union $\bigcup_{\mathbb{B}} I(a, b)$. Where for each (a, b) , $I(a, b)$ is a compact subinterval of $I(a) := [0, 1] \times \{a\}$;
- (ii) for each $(a, b) \in \mathbb{B}$, the restriction of g to $I(a, b)$ is a smooth diffeomorphism onto $I(b)$ satisfying $|Dg(t)| > 1$ for all t .

The *regular Cantor set* associated to g is the maximal invariant set

$$K = \bigcap_{n \geq 0} g^{-n} \left(\bigcup_{\mathbb{B}} I(a, b) \right).$$

Let $\Sigma_{\mathbb{B}}^+$ be the unilateral subshift associated to $\Sigma_{\mathbb{B}}$. There exists a unique homeomorphism $h: \Sigma_{\mathbb{B}}^+ \rightarrow K$ such that

$$h(\underline{a}) \in I(a_0), \text{ for } \underline{a} = (a_0, a_1, \dots) \in \Sigma_{\mathbb{B}}^+ \text{ and } h \circ \sigma = g \circ h,$$

where $\sigma^+: \Sigma_{\mathbb{B}}^+ \rightarrow \Sigma_{\mathbb{B}}^+$, is defined as follows $\sigma^+((a_n)_{n \geq 0}) = (a_{n+1})_{n \geq 0}$.

5.4 Expanding Maps Associated to a Horseshoe

Let Λ be a horseshoe associate a C^2 -diffeomorphism φ on the a surface M and consider a finite collection $(R_a)_{a \in \mathbb{A}}$ of disjoint rectangles of M , which are a Markov partition of Λ . Put the sets

$$W^s(\Lambda, R) = \bigcap_{n \geq 0} \varphi^{-n} \left(\bigcup_{a \in \mathbb{A}} R_a \right),$$

$$W^u(\Lambda, R) = \bigcap_{n \leq 0} \varphi^{-n} \left(\bigcup_{a \in \mathbb{A}} R_a \right).$$

There is a $r > 1$ and a collection of C^r -submersions $(\pi_a : R_a \rightarrow I(a))_{a \in \mathbb{A}}$, satisfying the following property:

If $z, z' \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$ and $\pi_{a_0}(z) = \pi_{a_0}(z')$, then we have

$$\pi_{a_1}(\varphi(z)) = \pi_{a_1}(\varphi(z')).$$

In particular, the connected components of $W^s(\Lambda, R) \cap R_a$ are the level lines of π_a . Then we define a mapping g^u of class C^r (expansive of type $\Sigma_{\mathbb{B}}$) by the formula

$$g^u(\pi_{a_0}(z)) = \pi_{a_1}(\varphi(z))$$

for $(a_0, a_1) \in \mathbb{B}$, $z \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$. The regular Cantor set K^u defined by g^u , describes the geometry transverse of the stable foliation $W^s(\Lambda, R)$. Analogously, we can describe the geometry transverse of the unstable foliation $W^u(\Lambda, R)$, using a regular Cantor set K^s define by a mapping g^s of class C^r (expansive of type $\Sigma_{\mathbb{B}}$).

Also, the horseshoe Λ is locally the product of two regular Cantor sets K^s and K^u . So, the Hausdorff dimension of Λ , $HD(\Lambda)$ is equal to $HD(K^s \times K^u)$, but for regular Cantor sets, we have that $HD(K^s \times K^u) = HD(K^s) + HD(K^u)$. Thus $HD(\Lambda) = HD(K^s) + HD(K^u)$ (cf. [PT93, chap 4]).

5.5 Intersections of Regular Cantor Sets and Property V

Let r be a real number > 1 , or $r = +\infty$. The space of C^r expansive maps of type Σ (cf. section 5.3), endowed with the C^r topology, will be denoted by Ω_{Σ}^r . The union $\Omega_{\Sigma} = \bigcup_{r > 1} \Omega_{\Sigma}^r$ is endowed with the inductive limit topology.

Let $\Sigma^- = \{(\theta_n)_{n \leq 0}, (\theta_i, \theta_{i+1}) \in \mathbb{B} \text{ for } i < 0\}$. We equip Σ^- with the following ultrametric distance: for $\underline{\theta} \neq \underline{\tilde{\theta}} \in \Sigma^-$, set

$$d(\underline{\theta}, \underline{\tilde{\theta}}) = \begin{cases} 1 & \text{if } \theta_0 \neq \tilde{\theta}_0; \\ |I(\underline{\theta} \wedge \underline{\tilde{\theta}})| & \text{otherwise} \end{cases},$$

where $\underline{\theta} \wedge \underline{\tilde{\theta}} = (\theta_{-n}, \dots, \theta_0)$ if $\tilde{\theta}_{-j} = \theta_{-j}$ for $0 \leq j \leq n$ and $\tilde{\theta}_{-n-1} \neq \theta_{-n-1}$.

Now, let $\underline{\theta} \in \Sigma^-$; for $n > 0$, let $\underline{\theta}^n = (\theta_{-n}, \dots, \theta_0)$, and let $B(\underline{\theta}^n)$ be the affine map from

$I(\underline{\theta}^n)$ onto $I(\theta_0)$ such that the diffeomorphism $k_n^\theta = B(\underline{\theta}^n) \circ f_{\underline{\theta}^n}$ is orientation preserving.

We have the following well-known result (cf. [Sul]):

Proposition. *Let $r \in (1, +\infty)$, $g \in \Omega_\Sigma^r$.*

1. *For any $\underline{\theta} \in \Sigma^-$, there is a diffeomorphism $k_n^\theta \in \text{Diff}_+^r(I(\theta_0))$ such that k_n^θ converge to k^θ in $\text{Diff}_+^{r'}(I(\theta_0))$, for any $r' < r$, uniformly in $\underline{\theta}$. The convergence is also uniform in a neighborhood of g in Ω_Σ^r .*
2. *If r is an integer, or $r = +\infty$, k_n^θ converge to k^θ in $\text{Diff}_+^r(I(\theta_0))$. More precisely, for every $0 \leq j \leq r-1$, there is a constant C_j (independent on $\underline{\theta}$) such that*

$$|D^j \log D [k_n^\theta \circ (k^\theta)^{-1}](x)| \leq C_j |I(\underline{\theta}^n)|.$$

It follows that $\underline{\theta} \rightarrow k^\theta$ is Lipschitz in the following sense: for $\theta_0 = \tilde{\theta}_0$, we have

$$|D^j \log D [k^{\tilde{\theta}} \circ (k^\theta)^{-1}](x)| \leq C_j d(\underline{\theta}, \tilde{\theta}).$$

Let $r \in (1, +\infty]$. For $a \in \mathbb{A}$, denote by $\mathcal{P}^r(a)$ the space of C^r -embeddings of $I(a)$ into \mathbb{R} , endowed with the C^r topology. The affine group $\text{Aff}(\mathbb{R})$ acts by composition on the left on $\mathcal{P}^r(a)$, the quotient space being denoted by $\overline{\mathcal{P}}^r(a)$. We also consider $\mathcal{P}(a) = \bigcup_{r>1} \mathcal{P}^r(a)$

and $\overline{\mathcal{P}}(a) = \bigcup_{r>1} \overline{\mathcal{P}}^r(a)$, endowed with the inductive limit topologies.

Remark 18. *In [MY01] is considered $\mathcal{P}^r(a)$ for $r \in (1, +\infty]$, but all the definitions and results involving $\mathcal{P}^r(a)$ can be obtained considering $r \in [1, +\infty]$.*

Let $\mathcal{A} = (\underline{\theta}, A)$, where $\underline{\theta} \in \Sigma^-$ and A is now an affine embedding of $I(\theta_0)$ into \mathbb{R} . We have a canonical map

$$\begin{aligned} \mathcal{A} &\rightarrow \mathcal{P}^r = \bigcup_{\mathbb{A}} \mathcal{P}^r(a) \\ (\underline{\theta}, A) &\mapsto A \circ k^\theta \in \mathcal{P}^r(\theta_0). \end{aligned}$$

Now assume we are given two sets of data $(\mathbb{A}, \mathbb{B}, \Sigma, g)$, $(\mathbb{A}', \mathbb{B}', \Sigma', g')$ defining regular Cantor sets K, K' .

We define as in the previous the spaces $\mathcal{P} = \bigcup_{\mathbb{A}} \mathcal{P}(a)$ and $\mathcal{P}' = \bigcup_{\mathbb{A}'} \mathcal{P}(a')$.

A pair (h, h') , ($h \in \mathcal{P}(a)$, $h' \in \mathcal{P}'(a')$) is called a *smooth configuration* for $K(a) = K \cap I(a)$, $K'(a') = K' \cap I(a')$. Actually, rather than working in the product $\mathcal{P} \times \mathcal{P}'$, it is better to go to the quotient Q by the diagonal action of the affine group $\text{Aff}(\mathbb{R})$. Elements of Q are called *smooth relative configurations* for $K(a), K'(a')$.

We say that a smooth configuration $(h, h') \in \mathcal{P}(a) \times \mathcal{P}'(a')$ is

- *linked* if $h(I(a)) \cap h'(I(a')) \neq \emptyset$;

- *intersecting* if $h(K(\underline{a})) \cap h'(K(\underline{a}')) \neq \emptyset$, where $K(\underline{a}) = K \cap I(\underline{a})$ and $K(\underline{a}') = K \cap I(\underline{a}')$;
- *stably intersecting* if it is still intersecting when we perturb it in $\mathcal{P} \times \mathcal{P}'$, and we perturb (g, g') in $\Omega_\Sigma \times \Omega_{\Sigma'}$.

All these definitions are invariant under the action of the affine group, and therefore make sense for smooth relative configurations.

As in previous, we can introduce the spaces $\mathcal{A}, \mathcal{A}'$ associated to the limit geometries of g, g' , respectively. We denote by \mathcal{C} the quotient of $\mathcal{A} \times \mathcal{A}'$ by the diagonal action on the left of the affine group. An element of \mathcal{C} , represented by $(\underline{\theta}, A) \in \mathcal{A}, (\underline{\theta}', A') \in \mathcal{A}'$, is called a relative configuration of the limit geometries determined by $\underline{\theta}, \underline{\theta}'$. We have canonical maps

$$\begin{aligned} \mathcal{A} \times \mathcal{A}' &\rightarrow \mathcal{P} \times \mathcal{P}' \\ \mathcal{C} &\rightarrow \mathcal{Q} \end{aligned}$$

which allow to define linked, intersecting, and stably intersecting configurations at the level of $\mathcal{A} \times \mathcal{A}'$ or \mathcal{C} .

Remark: For a configuration $((\underline{\theta}, A), (\underline{\theta}', A'))$ of limit geometries, one could also consider the *weaker* notion of stable intersection, obtained by considering perturbations of g, g' in $\Omega_\Sigma \times \Omega_{\Sigma'}$ and perturbations of $(\underline{\theta}, A), (\underline{\theta}', A')$ in $\mathcal{A} \times \mathcal{A}'$. We do not know of any example of expansive maps g, g' , and configurations $(\underline{\theta}, A), (\underline{\theta}', A')$ which are stably intersecting in the weaker sense but not in the stronger sense.

We consider the following subset V of $\Omega_\Sigma \times \Omega_{\Sigma'}$. A pair (g, g') belongs to V if for any $[(\underline{\theta}, A), (\underline{\theta}', A')] \in \mathcal{A} \times \mathcal{A}'$ there is a translation R_t (in \mathbb{R}) such that $(R_t \circ A \circ k^{\underline{\theta}}, A' \circ k'^{\underline{\theta}'})$ is a stably intersecting configuration.

5.6 Incompressible Sets

Next we will be considering some theorems that will be used in our arguments.

Definition 5. Let X_1, X_2 be metric spaces with metrics d_1 and d_2 respectively, a sequence of maps $f_i : X_1 \rightarrow X_2, i = 1, 2, \dots$ is said to be *uniformly bi-Lipschitz* if there exists a $C > 1$ such that

$$C^{-1}d_1(x, y) \leq d_2(f_i(x), f_i(y)) \leq Cd_1(x, y)$$

for all $x, y \in X_1$ and $i = 1, 2, \dots$; a $C > 1$ for which the relation holds is called a *uniform bi-Lipschitz constant* for the sequence.

Definition 6. A subset S of a metric space X is said to be *incompressible* if for any nonempty open subset Ω of X and any sequence f_i of uniformly bi-Lipschitz maps from Ω onto (possibly different) open subsets of X , the subset

$$\bigcap_{i=1}^{\infty} f_i^{-1}(S)$$

has the same Hausdorff dimension as X .

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